

18.306 Advanced Partial Differential Equations with Applications

G (P)

Prereq: 18.03 or 18.034; 18.04, 18.075, or 18.112

Units: 3-0-9

URL: <http://math.mit.edu/classes/18.306>

Lecture: TR11-12.30 (2-132)



Concepts and techniques for partial differential equations, especially nonlinear. Diffusion, dispersion and other phenomena. Initial and boundary value problems. Normal mode analysis, Green's functions, and transforms. Conservation laws, kinematic waves, hyperbolic equations, characteristics shocks, simple waves. Geometrical optics, caustics. Free-boundary problems. Dimensional analysis. Singular perturbation, boundary layers, homogenization. Variational methods. Solitons. Applications from fluid dynamics, materials science, optics, traffic flow, etc.

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Textbooks (Spring 2016)

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In my view, roughly speaking, this course is about:

- = learning PDE-theory to improve intuition, modeling skills, including numerical modeling.
- = improving analytical problem-solving skills.
- = improving PDE-background to assess more advanced Applied Math publications.
- = practicing filling-in gaps and making choices:
 - gaps = due to different student background;
 - gaps = in a problem's formulation;
 - >>> "research scrimmage" (in the sense of a simulated research).



Evaluation:

- == through homework assignments.
- == project at the end (letter grading).

① Introductory Concepts

uni-directional wave eqn.

$$\textcircled{X}_1 \quad u_t + cu_x = 0, \quad c > 0$$

$u(x_0) = f(x)$.

bi-directional wave eqn.

$$\textcircled{X}_2 \quad u_{tt} - c^2 u_{xx} = 0$$

$u(x_0) = f(x)$

$u_t(x_0) = g(x)$

$x - ct = \xi_1 \Leftrightarrow$ characteristic

slope $\frac{dx}{dt} = c \equiv \text{const.}$, speed of prop. of "information"

$u(x,t) = u(x(t),t) = u(\xi_1 + ct, t)$

↑ on a charact.

$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = cu_x + u_t = 0 \rightarrow u \text{ is constant along a charact.}$

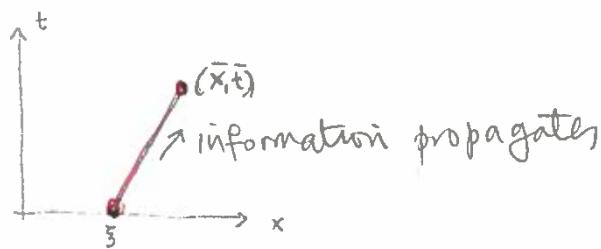
- Therefore

$u(x,t) = u(\xi_1 + ct, t) = f(\xi_1) = u(\xi_1, 0)$

$$\boxed{u(x,t) = f(\xi) = f(x-ct)} .$$

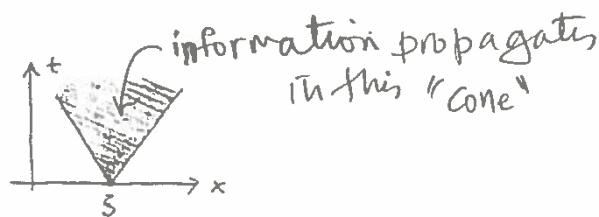
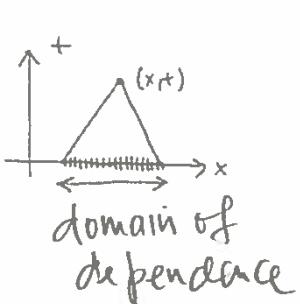
$f(x) \in C^1(\mathbb{R}) \Rightarrow u$ smooth solution of \textcircled{X}_1

- See all this in a more systematic fashion.
- Study \textcircled{X}_2 later.



domain of dependence of $u(\bar{x}, \bar{t})$ is ξ
 the influence of the initial value at ξ is felt
 along this charact.

For \otimes_2 we will see that



QUASI-LINEAR PDE

Consider this PDE in free space:

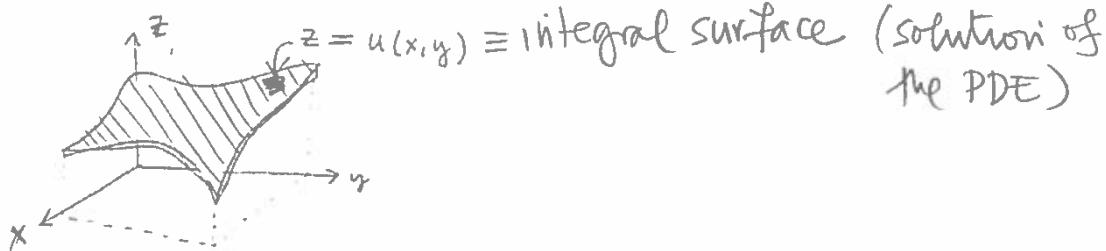
\otimes_3

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \quad u = u(x, y)$$

linear in higher
order terms.

Geometrical interpretation

3



vector field $\vec{t} = (a(x, y, u), b(x, y, u), c(x, y, u))^T$.

$\nabla \underbrace{s(x, y, z)}_{= \text{constant}} = \text{normal to the}$
 $\vec{t} \rightarrow \vec{t}$ int. tangent
 to integral surface

$\nabla(u(x, y) - z)$ int. surface.

In other words

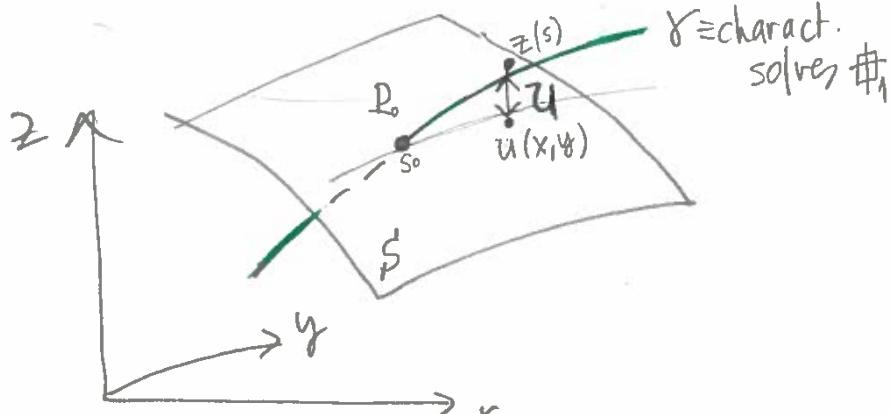
$$\frac{dx}{a(x, y, z)} = \frac{dy}{b(x, y, z)} = \frac{dz}{c(x, y, z)}$$

or choosing a parametrization:

ODEs: $\begin{cases} \frac{dx}{ds} = a(x, y, z) \\ \frac{dy}{ds} = b(x, y, z) \\ \frac{dz}{ds} = c(x, y, z) \end{cases}$

Characteristic equations #1 $a, b, c \in C^1(\Omega)$. (more than Lipschitz)

Let $P_0 = (x_0, y_0, z_0)$ a point where the charact. starts. initial data for ODE syst.



Let $P_0 = (x_0, y_0, z_0)$ be a point on the int. surface
 $\zeta: z = u(x, y)$. γ is a charact. passing through P_0 .
 Is γ entirely in S ? $\gamma = (x(s), y(s), z(s))$.
 Soln. of charact ODE.

Call

$$\left\{ \begin{array}{l} \boxed{u(s) = z(s) - u(x(s), y(s))} \\ \boxed{u(s) = 0} \text{ since } P_0 \in S. \end{array} \right.$$

Compute

$$\frac{dU}{ds} = \frac{dz}{ds} - u_x \frac{dx}{ds} - u_y \frac{dy}{ds} = c(x, y, z) -$$

$$-a(x, y, z)u_x - b(x, y, z)u_y = c(x(s), y(s), \boxed{U(s) + u(x(s), y(s))}) -$$

$$-a(x(s), \boxed{U(s) + u(x(s), y(s))})u_x - b(x(s), \boxed{U(s) + u(x(s), y(s))})u_y$$

ODE for $U \equiv$ mismatch

$\textcircled{2} s=s_0, \text{ get RHS} = \text{PDE}|_{s_0}$

$$\boxed{\frac{dU(s_0)}{ds} = 0, U(s_0) = 0} \xrightarrow{\text{uniqueness}} \boxed{U(s) \equiv 0}.$$

⑥ THE CAUCHY PROBLEM —

Generalization of our usual IVP.

for now let's
think of y
as time

Instead of initial data at say time $y = 0$
we have data on an initial curve Γ .

This curve Γ selects an integral surface
among candidates.

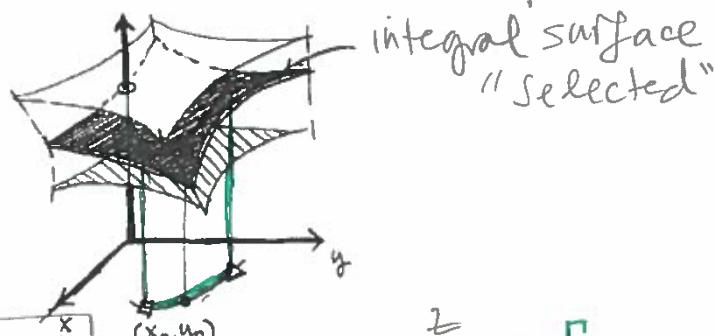
Let Γ be given parametrically:

$$x = f(s), \quad y = g(s), \quad z = h(s),$$

The solution of the PDE must satisfy

$$h(s) = u(f(s), g(s))$$

Schematic figure



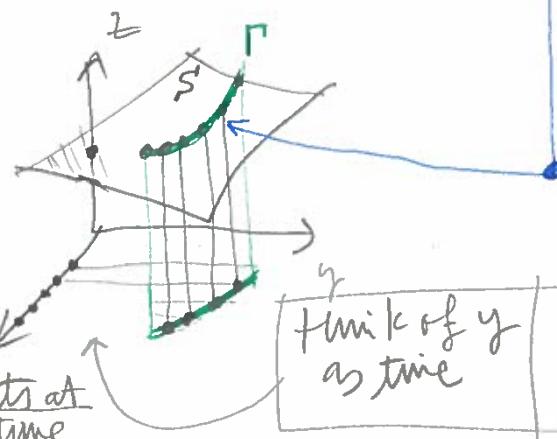
Cauchy problem

PDE + initial data

initial curve.

When $g(s) \equiv 0$ & $y \equiv t$ this is
the usual IVP

measurements at
different places & different time



function of y
as time

• Facts:

IVP: $\exists! P = (x, 0, h(x))$ which determines the solution

Cauchy problem: \exists several (infinitely many) P 's which lead to a specific solution.

Let $f(s), g(s), h(s) \in C^1(N(s_0))$, $N(s_0)$ = neighborhood of s_0 .

① How do we construct a solution?

Let $a(x, y, z), b(x, y, z), c(x, y, z) \in C^1(N(P_0))$

Consider the notation: $P_0 = (x_0, y_0, z_0)$.

$$x = \underline{X}(s, t), \quad y = \underline{Y}(s, t), \quad z = \underline{Z}(s, t)$$

↑ parameters

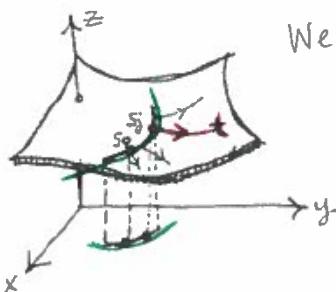
$$\left\{ \begin{array}{l} \frac{d\underline{X}}{dt} = a, \quad \underline{X}(s_0, t) = f(s) \\ \frac{d\underline{Y}}{dt} = b, \quad \underline{Y}(s_0, t) = g(s) \\ \frac{d\underline{Z}}{dt} = c, \quad \underline{Z}(s_0, t) = h(s) \end{array} \right.$$

Charact. ODEs near P_0 .

A 1-parameter family of ODES (in s)

ODE - finite dimensional problem — degrees of freedom
PDE - infinite "

Charact. ODEs - still infinite dimensional.



We have a 1-parameter family (in s)
of 3×3 systems of ODES.

Based on good properties of a, b and c we invoke the
existence and continuous dependence on (data) for
the solutions of the charac. eqns. #2:
initial data
- coef.
- parameters.

$$\Sigma(s, t), \Gamma(s, t), Z(s, t) \in C^1(N(s_0, t=0))$$

) The union of solutions of #2 will represent an integral surface.

$$\Sigma : z = u(x, y) = Z(s, t)$$

we want \uparrow \uparrow not this
this form one.

Use the first 2 solutions of system #2 to get

$$s = S(x, y), t = T(x, y) \quad (\times)$$

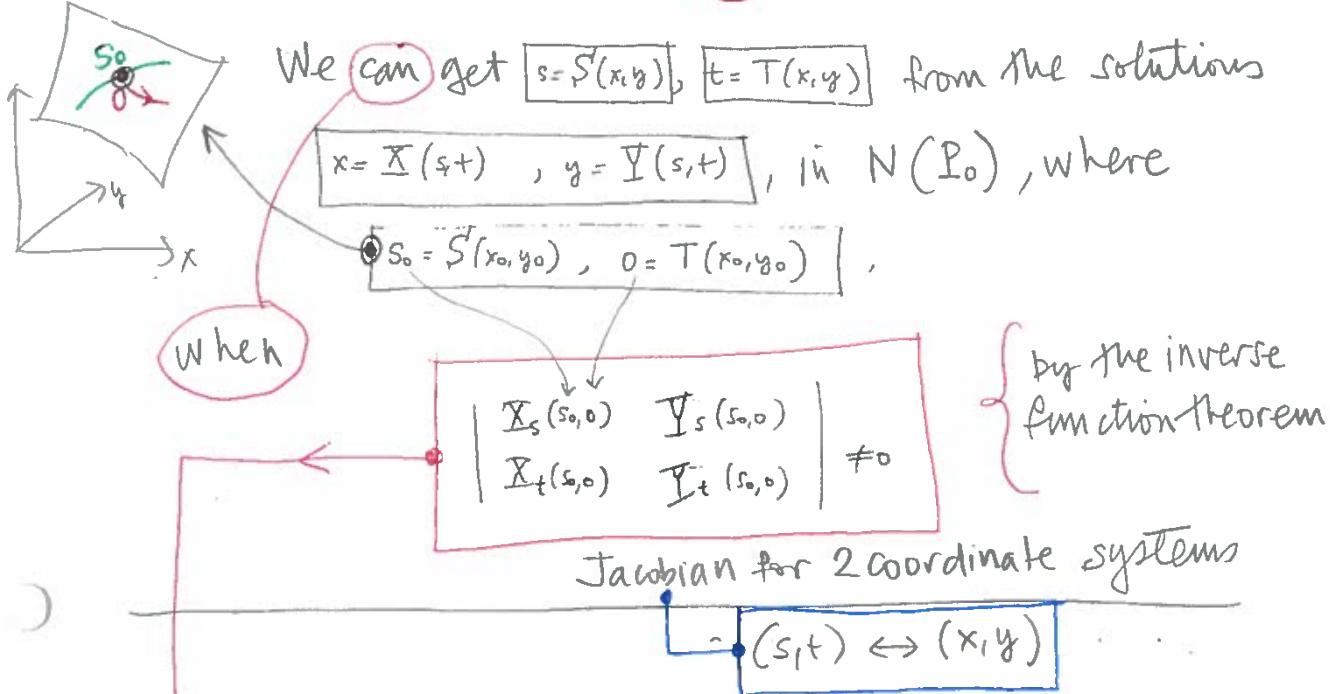
so that

$$u(x, y) = Z(S(x, y), T(x, y))$$

Integral surface is expressed as

$$\Sigma: z = \Sigma(s(x,y), t(x,y))$$

Need to check when step (*) above is valid:



therefore the important condition is

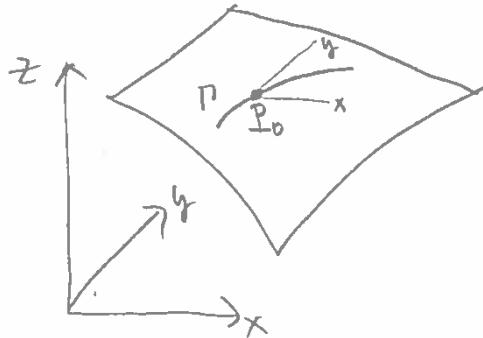
$$\begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{vmatrix} \neq 0$$

data

"initial" ODE field.

which allows us to have expressions (*),
namely inverting the roles of (s,t) and (x,y) .

- Interpretation when $J = \begin{vmatrix} f' & g' \\ a & b \end{vmatrix}_{P_0} = 0$



At the point P_0 we have that

$$\left\{ \begin{array}{l} J = bf' - ag' = 0 \\ h(s) = u(f(s), g(s)) \rightarrow h'(s_0) = f'u_x + g'u_y \\ \hline c = au_x + bu_y \end{array} \right. \quad \begin{array}{l} \text{(data)} \\ \text{"initial" PDE} \end{array}$$

Eliminate u from equations

$$\left\{ \begin{array}{l} bh' - cg' = 0 \\ ah' - cf' = 0 \\ \hline bf' - ag' = 0 \end{array} \right. \Leftrightarrow (a, b, c) \times (f', g', h') = 0$$

\Rightarrow Γ is characteristic!

Also $\begin{bmatrix} f' & g' \\ a & b \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} h' \\ c \end{bmatrix}$

does not have a solution.

We have $\frac{\partial u}{\partial s}$ along Γ but cannot get any other directional derivative using u_x and u_y .

\Rightarrow Stuck on initial curve.

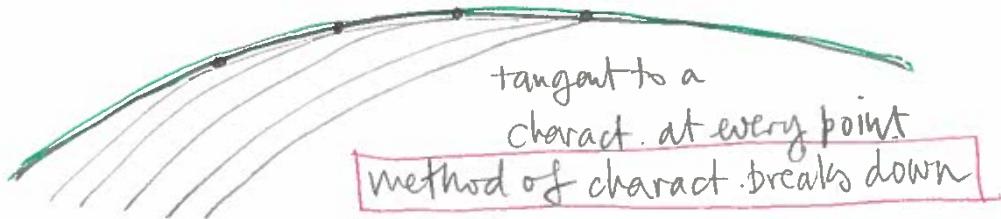
\Rightarrow Power series (local) solution will not work.

① Facts

very likely not

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- Arbitrary data on a characteristic $\underbrace{\text{may not be}}_{\text{compatible with the equations (PDE/charac)}}$
- This problem can happen on caustics when characteristics cross (as in seismic problem shown).
Caustic = envelope of characteristics.



②

Linear PDE, as a special case

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$$

↑ forcing term.

$$\left\{ \begin{array}{l} \frac{dx}{dt} = a(x,y) \\ \frac{dy}{dt} = b(x,y) \end{array} \right\} \text{ independent of solution}$$

$$\Rightarrow \frac{du(x(t), y(t))}{dt} = G(t)u + D(t)$$

"PDE" in a single
"variable"
(parameter.)

$$\left\{ \begin{array}{l} G(t) = c(x,y); X(t), Y(t) \\ D(t) = d(x,y) \end{array} \right.$$

When convenient (with $b(x,y) \neq 0$) can be put in the form

$$\frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}$$

a/b can be thought as a variable wave speed

EXAMPLE

The unidirectional WAVE EQN (model wave egn.)

①

$$\begin{cases} u_y + cu_x = 0, \quad c = \text{const.} \\ u(x, 0) = h(x) \end{cases}$$

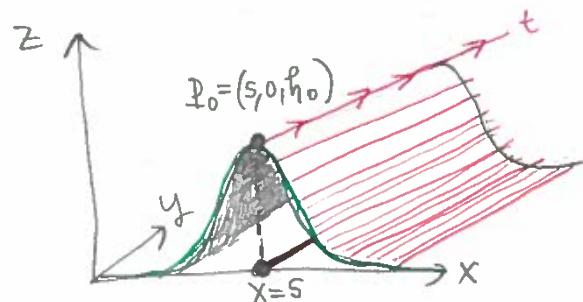
y plays the role of time! ←

IVP

$$\Gamma = (x = f(s), y = g(s), z = h(s)) = (s, 0, h(s))$$

Charac. ODES. $\Rightarrow \begin{cases} \frac{dx}{dt} = c \Rightarrow x = X(s, t) = \sqrt{x_0 + ct} = s + ct \\ \frac{dy}{dt} = 1 \Rightarrow y = Y(s, t) = \sqrt{y_0 + t} = 0 + t = t \\ \frac{dz}{dt} = 0 \Rightarrow z = Z(s, t) = z_0 = h(s) \end{cases}$

For each s we have a system of ODEs.
We have a "continuum" of ODEs to be solved
"in parallel".



Final Solution

$$z = u(x, y) = h(s), \text{ where } s = \underline{s}(x, y),$$

we conclude that $s = \underline{s}(x, y) = x - cy$

and

$$u(x, y) = h(x - cy)$$

↑ speed of propagation
travelling wave with profile $h(x)$

(2)

Burgers Eqn.

$$\rightarrow u_t + uu_x = \mu u_{xx} \quad (\text{"viscous Burgers"})$$

$$u_t + uu_x = 0 \quad (\text{"inviscid Burgers"})$$

In the original goursat form, notation:

$$\begin{cases} u_y + uu_x = 0 \\ u(x_{10}) = h(x) \end{cases}$$

$$\begin{cases} x = s \\ y = 0 \\ z = h(s) \end{cases} \quad \text{initial curve} \Rightarrow \text{IVP}$$

Charact. ODES

$$\begin{cases} \frac{dx}{dt} = z & \text{time indep. along charac.} \\ \frac{dy}{dt} = 1 \\ \frac{dz}{dt} = 0 \end{cases} \Rightarrow \begin{aligned} x &= x_0 + zt = \textcolor{red}{(s)} + zt \\ y &= y_0 + t = \textcolor{red}{(0)} + t \\ z &= z_0 = h(s) \end{aligned}$$

easy to invert.

$$s = x - zy, \quad t = y, \quad z = h(s)$$

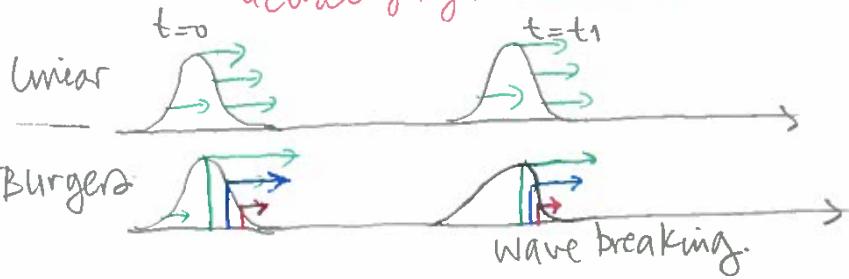
$$z = u(x, y) = h(x - \textcolor{red}{uy}), \quad \text{Solution in implicit form.}$$

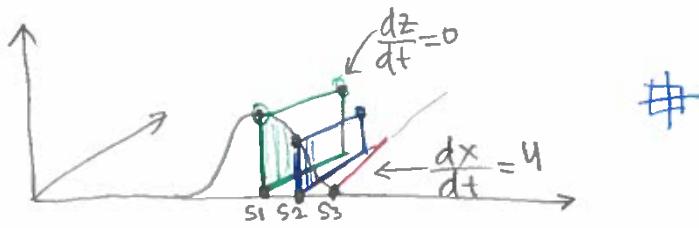
Compare with linear solution

*speed determined by solution,
actually by initial data.*

$$u(x, y) = h(x - cy)$$

pre-determined speed





• Note: Linear = "passive transport". Super simple example is the transport of a tracer.

$$C_t + U C_x = K C_{xx}, \quad U = \text{constant wind speed}$$

$C(x,t) = \text{concentration of pollutant}$

$K = \text{diffusion coefficient (for pollutant dispersion)}$

$x=0$ $C_0(x)$

• Note: Wave breaking such as

$t=0$ $t=t_1$ $t=t_2$

multi-valued profile. need a more sophisticated model

• Note at a given s along Γ

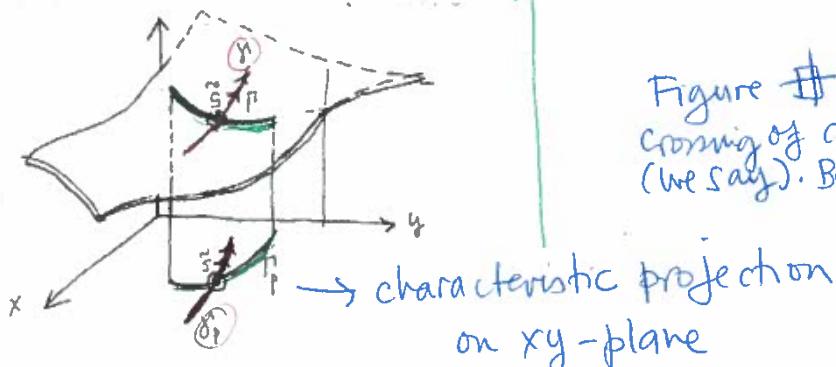
$$\begin{cases} x(s,t) = x_0(s) + z(s)t = f(s) + h(s)t \\ y(s,t) = y_0(s) + t = g(s) \end{cases} \quad \text{EX2}$$

explicit inversion

$S = S(x,y)$ can be
 $T = T(x,y)$ harder
in some cases

$$\delta_p(t) = (x(\tilde{s},t), y(\tilde{s},t)),$$

Figure # crossing of characteristics (we say). But in actually



Charac ODEs

$$\begin{cases} \frac{dx}{dt} = a \\ \frac{dy}{dt} = b \\ \frac{dz}{dt} = c \end{cases}$$

- Take the solution of Burgers eqn

$$u = h \underbrace{(x - uy)}_s,$$

When does derivative "blow up"?

$$\left\{ \begin{array}{l} u_x = h' \frac{\partial s}{\partial x} = h'(s) (1 - u_x y) \\ u_x (1 + h'(s)y) = h'(s) \end{array} \right.$$

$$u_x = \frac{h'(s)}{1 + h'(s)y}$$

$$y = t_c = -\frac{1}{h'(s_0)};$$

↑
critical time

$$h'(s) < 0$$

monotonically increasing initial profile does not blow up.

- t_c associated with $\min(h'(s))$, namely a negative derivative at the point of max. slope.

- At $t \geq t_c$: Solution does NOT exist in the classical sense. Have to re-interpret equation and solution to provide some meaning: "weak solutions"
- Smooth data $h(s)$ and the solution does not even have the first derivative: loss of regularity

A first step in dealing with a weak solution.

Suppose we have a PDE in divergence form such as

$$\nabla \cdot (S(u) R(u))^T = \frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0.$$

which leads to

$$R' u_y + S' u_x = 0.$$

We get Burgers equation if $S'(u) = u R'(u)$: $R' u_y + R' u u_x = 0$
 $\nabla \cdot (u, u^2/2) = 0$
 $u_y + u u_x = 0$

Divergence form leads to a Balance eqn
or, a Conservation Law:

$$\frac{d}{dy} \int_a^b R(u(s,y)) ds = [S(u(a,y)) - S(u(b,y))].$$

total "mass"
of this object $R(u)$

is balanced by the
effect of $S(u)$ at
the boundary.

Conservation of total
"mass of $R(u)$ " when
this is zero

- In some fluid problems it is total MASS indeed

- Burgers: $\frac{d}{dt} \int_a^a u(x,t) dx = \frac{u^2(a,t) - u^2(a,t)}{2}, a \gg 1$

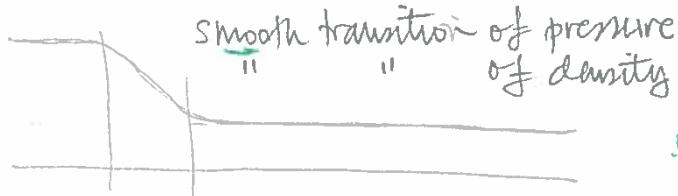
$v = \text{density} = \frac{\text{mass}}{[L]}$



- Discontinuity in Burgers:



Example: tidal bore
running upstream on a river.



leads to



Jump = shock (will see some more later)

length scale is so small we model collapsed to a point

① Problem: If a jump forms (like in Burgers) we have no more a classical solution. Can the PDE tell us how the jump will propagate?

- Yes! Use the integral form of the Conserv. law:

$$a < \xi(y) < b$$

jump position
in "time".

Leibnitz integral rule

$$\begin{aligned} 0 &= S(u(b,y)) - S(u(a,y)) + \\ &+ \frac{d}{dy} \left\{ \int_a^{\xi(y)} R(u) dx + \int_{\xi(y)}^b R(u) dx \right\} = S(u(b,y)) - S(u(a,y)) + \\ &+ \xi'(y) R(u^-) - \xi'(y) R(u^+) + \int_a^{\xi(y)} \frac{\partial R}{\partial y}(u) dx + \int_{\xi(y)}^b \frac{\partial R}{\partial y}(u) dx = \\ &= S(u(b,y)) - S(u(a,y)) - \frac{d\xi}{dy} (R(u^+) - R(u^-)) \end{aligned}$$

use PDE

$$\ominus S(u^-) + S(u(a,y)) - S(u(b,y)) + S(u^+)$$

$$\Rightarrow \text{shock speed} = \frac{d\xi}{dy} = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$

SHOCK CONDITION

Note: depends on

divergence form adopted.

Rankine-Hugoniot

Condition

$$\text{Ex: } u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad R(u) = u, \quad S(u) = u^2/2$$

$$\text{Burgers: } (u^2/2)_t + (u^3/3)_x = 0, \quad R(u) = u^2/2, \quad S(u) = u^3/3$$

Comments .

- "to solve a PDE" can be subtle
 - classic solution
 - weak solution
- informal notion of well-posed problems.
we want some desirable features:
 - (a) problem HAS a solution - EXISTENCE
 - (b) solution is unique
 - (c) solution depends continuously on data:
 - initial and / or boundary data
 - coefficients; forcing term ...
- informal → not being specific
- regularity : would be nice PDE of order k
to have all solutions in C^k
(derivatives of order k are continuous)
- Conservation law is well-posed if we allow for
properly defined
generalized/weak solutions
in a wider class of solutions
(appropriate)
- Structure of PDE \Rightarrow force us to abandon notion of
classical solution
- regularity study of weak solutions can sometimes
lead to a classical solution .

A step in more generality. — NON-LINEAR PDE.

GENERAL FIRST ORDER PDE in 2 VARIABLES

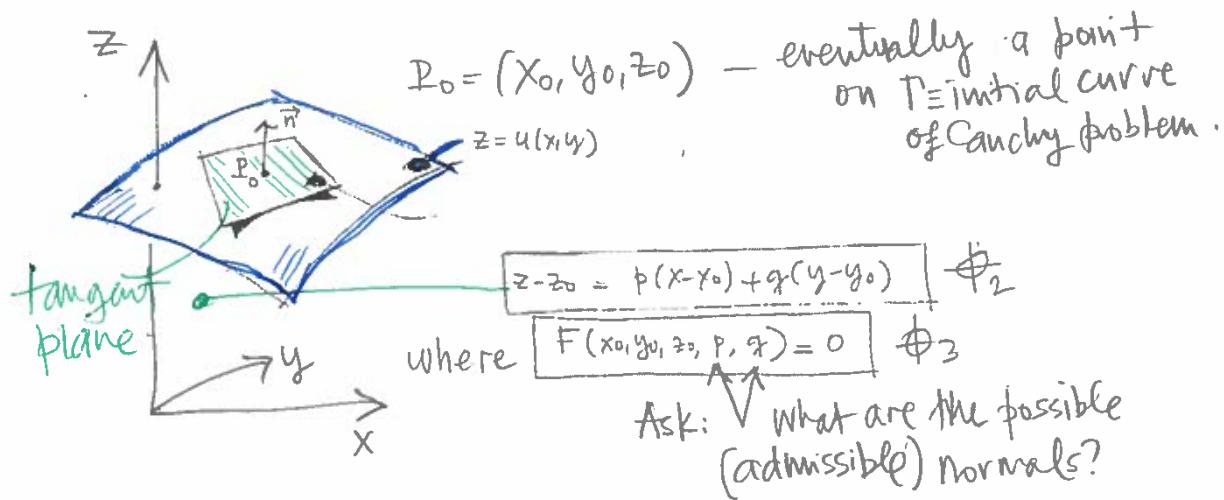
nonlinear
PDE

$$\Phi_1 \quad F(x, y, u, u_x, u_y) = 0$$

Call
 $z = u(x, y)$
 $p = u_x$
 $q = u_y$
 $F \in C^2$

How to extract a geometrical picture out of this PDE?

$z = u(x, y)$ still the integral surf. in 3D
 $(u_x, u_y, -1) = \vec{n}$ to the int. surface.
 Call $(p, q, -1) = (u_x, u_y, -1)$.

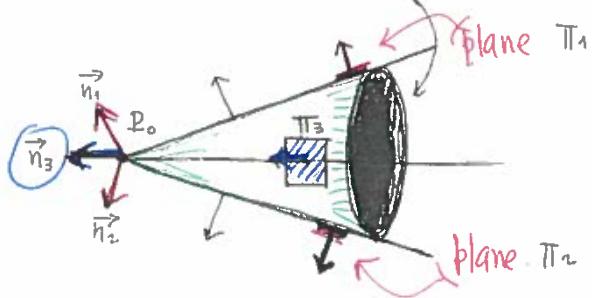


Note:

- (1) We fixed P_0 and PDE at P_0 (\oplus_3) \Rightarrow 1 parameter family of tangent planes.

p and q coupled thru #3

- (2) Family of normal $\vec{n} = (p, q(p), -1)$ all "attached" to P_0 . Envelope (tangent to) of all these planes is a **CONE**:



In other words (locally) our problem became

$$\begin{array}{ll} \#_A & \left\{ \begin{array}{l} F(x_0, y_0, z_0, p, q) = 0 \\ H(x, y, z, p, q) = 0 \end{array} \right. & \text{("PDE at } P_0\text{")} \\ \#_B & & \text{(Family of planes at } P_0\text{)} \end{array}$$

where $H = z - z_0 - p(y - x_0) - q(p)(y - y_0) = 0$

↑ extracted (explicitly when possible)
from #_A above.

Condition for the envelope

$$\frac{\partial H}{\partial p} = -(x-x_0) - \frac{dq}{dp}(y-y_0) = 0$$

Using $q(p)$ we can get $p = p(x, y)$ and characterize these MONGE CONES.

• EX:

$$\text{PDE: } -4xy = 1$$

$$F = pq - 1,$$

therefore

$$q = \frac{1}{p} \quad \& \quad \frac{dq}{dp} = -\frac{1}{p^2}$$

$$\begin{aligned} \frac{dq}{dp} &= -\frac{(x-x_0)}{(y-y_0)} = -\frac{1}{p^2} && \text{envelope} \quad \text{PDE} \\ \frac{1}{p^2} &= q^2 = \frac{(x-x_0)}{(y-y_0)} \end{aligned}$$

Substituting in #B

$$z - z_0 - \left(\frac{1}{p}\right)(x-x_0) - \left(\frac{1}{p}\right)(y-y_0) = 0$$

1 parameter family
of planes

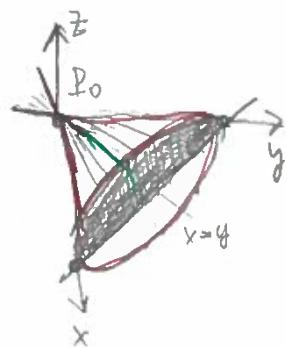
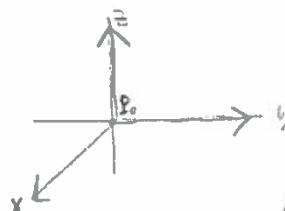
$$z - z_0 - \sqrt{(x-x_0)(y-y_0)} - \sqrt{(x-x_0)(y-y_0)} = 0$$

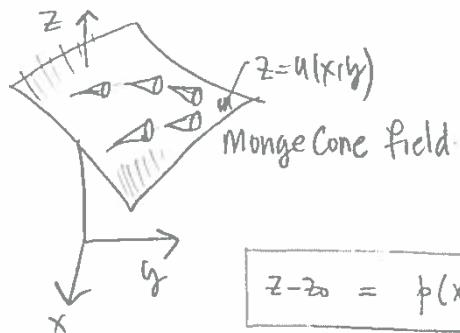
$$z - z_0 = 2\sqrt{(x-x_0)(y-y_0)}$$

$$(z-z_0)^2 = 4(x-x_0)(y-y_0)$$

$$\Rightarrow u(x, y) = z_0 + \sqrt{4(x-x_0)(y-y_0)}$$

$$\text{WLOG: } (x_0, y_0, z_0) = (0, 0, 0) \Rightarrow z^2 = 4xy$$





Back to our problem

Family of planes
 $H(x, y, z, p, q) = 0$

$$z - z_0 = p(x - x_0) + q(y - y_0), \text{ where } p = u_x, q = u_y,$$

and $q = q(p)$ (thru PDE)

$$F(x_0, y_0, z_0, p, q) = 0$$

$$\vec{n}_p = (p, q(p), -1)$$

① Formulate the characteristic system of ODES

envelope condition $\Rightarrow \left\{ \begin{array}{l} H(x, y, z, p, q) = 0 \\ \frac{\partial H}{\partial p}(x, y, z, p, q) = 0 \end{array} \right. \Leftrightarrow -(x - x_0) - \frac{dq}{dp}(y - y_0) = 0$ family of planes

- To obtain $\frac{dq}{dp}(p)$ use PDE:

$$\frac{dF}{dp} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q} \frac{dq}{dp} = 0 \rightarrow \frac{dq}{dp} = - \frac{F_p(x_0, y_0, z_0, p, q)}{F_q(x_0, y_0, z_0, p, q)} \quad (**)$$

Substitute in envelope condition.

$$-(x - x_0) + \frac{F_p}{F_q}(y - y_0) = 0$$

$$\frac{(x - x_0)}{F_p} = \frac{(y - y_0)}{F_q}$$

First 2
charact eqns.
will follow...

(*)

$$\frac{F_p}{F_q} = \frac{(x - x_0)}{(y - y_0)}$$

Substitute in tangent plane expression:

$$\boxed{\frac{z-z_0}{y-y_0} = \frac{p(x-x_0)}{(y-y_0)} + q(p)} = p \frac{F_p}{F_q} + q(p) = \frac{pF_p + qF_q}{F_q}$$

and get

$$\boxed{\frac{z-z_0}{pF_p + qF_q} = \frac{y-y_0}{F_q}}. \quad (*_2)$$

④ $(*_1) + (*_2)$

$$\boxed{\frac{z-z_0}{pF_p + qF_q} = \frac{y-y_0}{F_q} = \frac{x-x_0}{F_p}}$$

See $(**)$

- all evaluated at P_0

- $p, q(p)$

- (x_1, y_1, z)

point on plane II near P_0 .

$$\boxed{\frac{dz}{pF_p + qF_q} = \frac{dy}{F_q} = \frac{dx}{F_p}}$$

④ Incomplete system of characteristic ODES.

#

$$\boxed{\begin{aligned}\frac{dx}{ds} &= F_p \\ \frac{dy}{ds} &= F_q \\ \frac{dz}{ds} &= pF_p + qF_q\end{aligned}}$$

double check for quasi-linear
 \hookrightarrow pg 20.

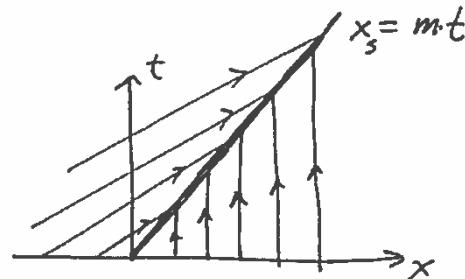
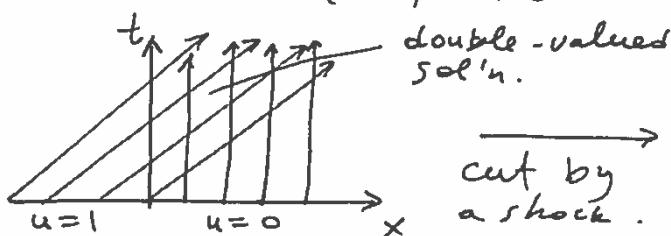
UNIT PART II

Lecture 24: Weak solns of hyperbolic PDE,
Shocks, rarefactions, Rankine-Hugoniot cond's., Entropy cond's.
Traffic flow. 24.1

Weak solns.

Ex. $u_t + u u_x = 0 \quad x \in \mathbb{R}, t > 0$

$$u(x, 0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

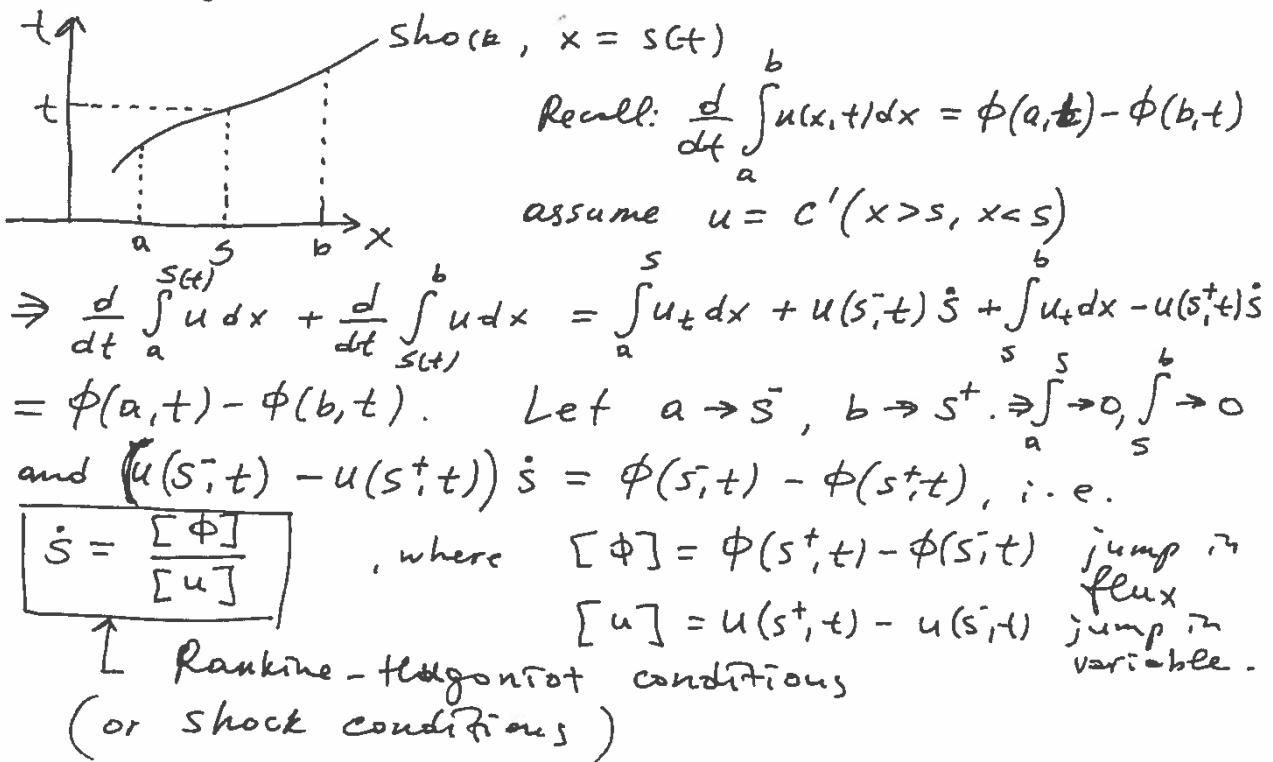


The idea is to get rid of the troublesome region by putting a discontinuity

Q: What is m ? A: Rankine-Hugoniot conditions

Q: Is the soln unique? A: Entropy conditions.

Note: PDE is derived from integral conservation laws assuming smooth solns. Now the assumption is no longer valid. To fix things, go back to integral conservation laws.



24.2

For Burgers' eqn, written as; $u_t + (u^2/2)_x = 0$,
 the flux is $\phi = u^2/2$. Then $s = m = \frac{[u^2/2]}{[u]} = \frac{1}{2} \frac{u^+^2 - u^-^2}{u^+ - u^-} = \frac{1}{2} (u^+ + u^-) = \frac{1}{2}$.

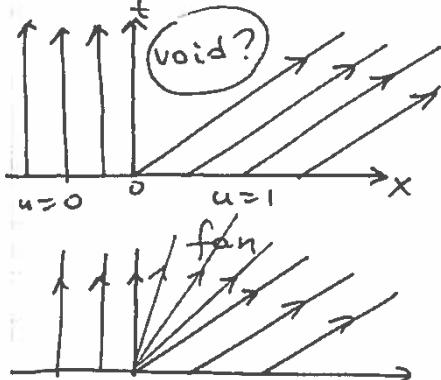
Note: For smooth solns, $u_t + uu_x = 0 \Rightarrow$ equivalent to $(u^2/t + \frac{2}{3}u^3)_x = 0$.

Then $s = \frac{[2/3 u^3]}{[u^2]} = \frac{0 - 2/3}{0 - 1} = \frac{2}{3} \neq \frac{1}{2}$!

Moral: Different forms of conservation laws give different weak solns.
 Which form is correct is decided by the physics.

Rarefactions.

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$



Q: What is the soln in $0 < x < t$?

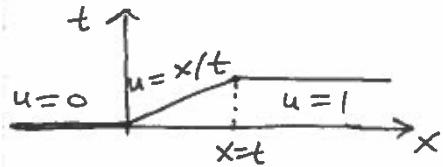
Idea: Smooth out the initial jump.
 Then char's will turn smoothly from $x=0$ to $x=t$



From $u_t + uu_x = 0$, it

follows that $u = \text{const} = \alpha$ on $\frac{dx}{dt} = \alpha$, i.e. on $x = \alpha t$.
 Then in the fan (rarefaction wave), the soln is

$u = \alpha = x/t \Rightarrow$ at $t > 0$, the soln looks like:



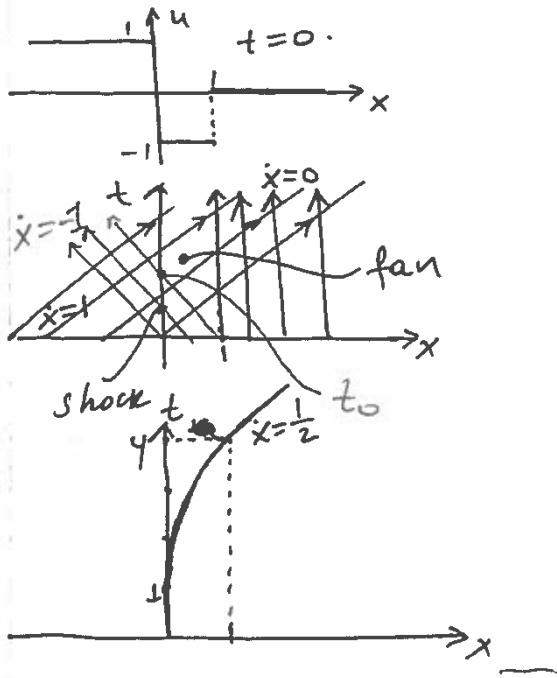
$$\Rightarrow f' \cdot (u - x/t) = 0, \Rightarrow \boxed{(u = x/t)}$$

Note: $u = x/t$ is a similarity soln, i.e. $u = f(\xi)$, $\xi = x/t$

$$u_t = f' \cdot (-\frac{x}{t^2}), \quad u_x = f' \frac{1}{t}$$

24.3

Ex. Solve $u_t + uu_x = 0$ with $u(x, 0) = \begin{cases} 1, & x < 0 \\ -1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$



Expect a fan where char's diverge and a shock where they collide.

when char's $x=1$ and $x=-1$ collide, a shock forms with speed $s = \frac{1}{2}(u_+ + u_-) =$

\Rightarrow up until $t = t_0$, which corresponds to the rightmost characteristic from $x = -1$ intersecting $x = 1$,

the shock sits at $x = 0$.

After $t = t_0$, the shock path is determined by $s = \frac{1}{2}(u_+ + u_-)$, where $u_- = 1$ as before, but u_+ is the state in the fan rather than $u = -1$.
 \Rightarrow we need to find t_0 and the fan sol'n next.
 $t_0 = 1$ as easily seen. The fan state is found from $u = \text{const} = \alpha$ on $x-1 = \alpha t$.

$$\Rightarrow u = \frac{x-1}{t} \leftarrow \text{fan}.$$

$$\Rightarrow \text{at } t > t_0 = 1, \quad s = \frac{1}{2} \left(1 + \frac{s-1}{t} \right)$$

This ODE with $s(1) = 0$ is solved by $s(t) = t + 1 - 2\sqrt{t}$

The sol'n holds only up to $t = t_1$ corresponding to the ~~exit~~ exit out of the fan that takes place at $s = 1$, i.e. $t_1 - 2\sqrt{t_1} = 0 \Rightarrow t_1 = 4$.

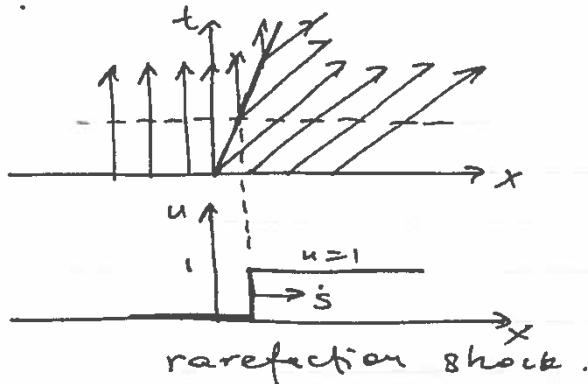
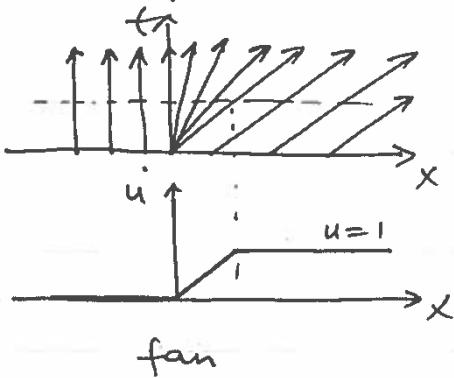
After that, $s = \frac{1}{2}(1 + 0) = \frac{1}{2}$ and $s = s(4) + \frac{1}{2}(t-4) = \frac{t}{2} - 1$.

24.4

Entropy cond'ns.

Consider $u_t + uu_x = 0$, $u(x, 0) = \begin{cases} 0, & x < 0 \\ u_-, & x > 0. \end{cases}$

Two possible sol'n's:



$$\dot{s} = \frac{1}{2}(u_+ + u_-) = \frac{1}{2}$$

Q: Which one is the correct sol'n?

A: A unique sol'n satisfies entropy cond'ns.

Lax entropy cond'n: $f'(u_+) < \dot{s} < f'(u_-)$,

where $f: u_t + (f(u))_x = 0$.

That is, the ^{Lax} shock overtakes characteristics ahead and is overtaken by characteristics from behind. In gasdynamic terms, a shock is supersonic with respect to the state ahead and subsonic wrt state behind.

For Burgers' eqn, $f = u^2/2$, \Rightarrow Lax cond'n is:

$$\boxed{u_+ < \dot{s} < u_-}$$

\Rightarrow If $u_- < u_+$, we can only have fans, no shocks.

Traffic flow:

24.5

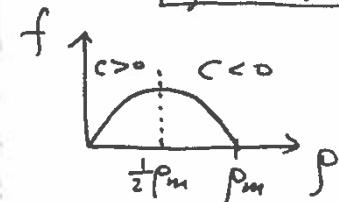
One lane, no exits/entrances. 

- $\rho(x,t)$ - number of cars per unit length of the road.
- $f(x,t)$ - # of cars crossing a given section of the road per unit time, flux.

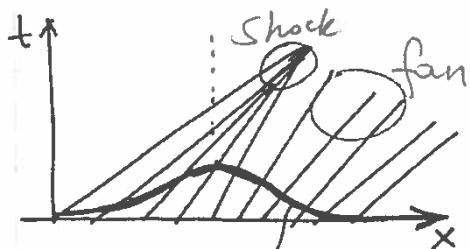
$f = \rho \tilde{u}$, where $\tilde{u} = \tilde{u}(\rho)$ is the "desired velocity".

Conservation of cars: $\frac{d}{dt} \int_{x_1}^{x_2} \rho dx = f(x_1,t) - f(x_2,t)$

$$\boxed{\rho_t + f_x = 0}$$



generic shape of f .
(fundamental diagram)



initial profile of ρ . char's: $\frac{dx}{dt} = c(\rho)$

Typically, $\tilde{u} = u_0(1 - \rho/\rho_m)$

Then $f = u_0 \rho (1 - \rho/\rho_m)$
and $c = f' = u_0 (1 - 2\rho/\rho_m)$

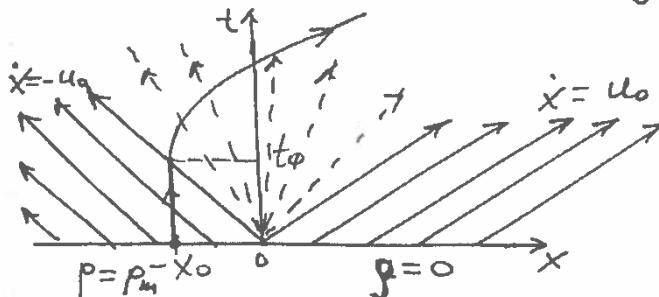
$$\Rightarrow \boxed{\rho_t + c(\rho) \rho_x = 0}$$

solve by MOC.

$$\begin{cases} \rho(x,t) = \rho(\xi,0) \\ x = \xi + c(\rho)t \end{cases} \quad \text{solv}$$

24.6

Ex. Traffic motion starting from the green light.



$$pt + C(\rho)\rho_x = 0$$

$$\rho(x, 0) = \begin{cases} p_m, & x < 0 \\ 0, & x > 0 \end{cases}$$

$$C(\rho) = u_0 \left(1 - \frac{2\rho}{\rho_m}\right) = \frac{at}{t_c}$$

$$= \begin{cases} u_0, & x > 0 \\ -u_0, & x < 0 \end{cases}$$

The fan between $x = u_0 t$ and $x = -u_0 t$ is given by $\rho = \text{const} = \alpha$ on $x = \alpha t = C(\rho)t \Rightarrow$

$$C(\rho) = u_0 \left(1 - \frac{2\rho}{\rho_m}\right) = x/t \quad \text{and}$$

$$\boxed{\rho = \frac{p_m}{2} \left(1 - \frac{x}{u_0 t}\right)} \quad \text{fan}$$

The car initially at $x = -x_0 < 0$ starts moving at $t = t_0$, when its path hits the fan.

$t_0 = x_0/u_0$. \Rightarrow If $x = z(t)$ describes the car path, then $z = -x_0$ at $0 \leq t \leq t_0$.

At $t > t_0$, the path is described by the equation of motion

$$\frac{dz}{dt} = u = u_0 \left(1 - \frac{\rho}{\rho_m}\right) \Big|_{x=z} = \frac{u_0}{2} \left(1 + \frac{z}{2u_0 t}\right)$$

(using the sol'n in the fan).

We need to solve this ODE subject to $z(t_0) = -x_0$.

The sol'n: $z(t) = \begin{cases} -x_0, & 0 \leq t \leq x_0/u_0 \\ u_0 t - 2\sqrt{u_0 x_0 t}, & t > x_0/u_0 \end{cases}$

Note 1: The time it takes to pass the light ($z(t_p) = 0$), $\therefore t_p = \frac{4x_0}{u_0}$

Note 2: $z < u_0$ at all $t > t_0$.

24.7

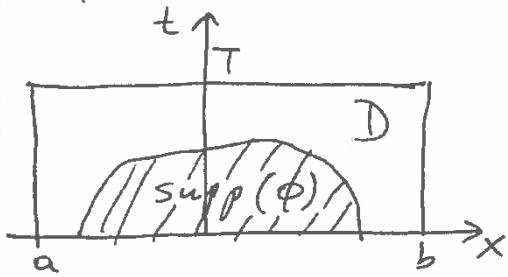
Formal def. of weak solns of $u_t + f_x = 0$.

$$\textcircled{*} \quad \boxed{u_t + (f(u))_x = 0}, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}, t > 0.$$

Classical solns are $C^1(\mathbb{R})$ (both in x and t).

Let $f, u \in C^1(\mathbb{R})$, and $\phi \in C^1(\mathbb{R})$
 (more precisely, $C^1(\mathbb{R} \times \mathbb{R}_+)$ where $\mathbb{R}_+ = [0, \infty)$)
 $\qquad \qquad \qquad x \in \mathbb{R}$

Assume ϕ has compact support, $\text{supp}(\phi)$
 let D contain $\text{supp}(\phi)$.



Next, multiply $u_t + f_x = 0$ by ϕ
 and integrate over D .
 Then integrate by parts
 to move derivatives over to ϕ

$$\begin{aligned} & \int_0^T \int_a^b (\phi u_t + \phi f_x) dx dt = 0 = \\ &= \int_a^b dx \left[\phi u \Big|_0^T - \int_0^T u \phi_t dt \right] + \int_0^T dt \left[\phi f \Big|_a^b - \int_a^b f \phi_x dx \right] = \\ &= - \int_a^b u(x, 0) \phi(x, 0) dx - \int_a^b \int_0^T u \phi_t dx dt - \int_0^T \int_a^b f \phi_x dx dt. \end{aligned}$$

\Rightarrow The weak form of $\textcircled{*}$ is:

$$\textcircled{**} \quad \boxed{\int_0^T \int_a^b (u \phi_t + f \phi_x) dx dt + \int_a^b u_0(x) \phi(x, 0) dx = 0}$$

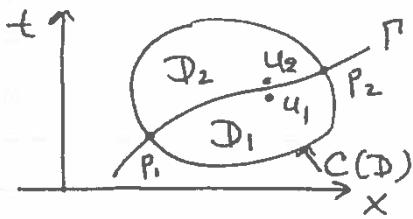
\Rightarrow if u solves $\textcircled{*}$ and ϕ is a test function, then
 u satisfies $\textcircled{**}$. Note that $\textcircled{**}$ makes sense
 even if u, f are not C^1 functions.

This motivates the following definition.

(24.8)

Def: A bounded, p-w smooth $u(x, t)$ is a weak soln of $u_t + f_x = 0$, $u(x, 0) = u_0(x)$ iff u solves $\star\star$ for any test function ϕ .

Q: Does this definition recover Rankine-Hugoniot conditions?



Consider a curve Γ across which u has a discontinuity. Let $\Gamma \in D$ and $D_1 \cap D_2 = \emptyset$. Let the test function ϕ be s.t. $\phi = 0$ on $C(D)$.

$$\Rightarrow 0 = \int_D (u\phi_t + f\phi_x) dx dt = \int_{D_1} + \int_{D_2}$$

$$\int_{D_1} = \int_{D_1} [(u\phi)_t + (f\phi)_x] dx dt = \int_{D_1} (-u\phi dx + f\phi dt) =$$

(by Green's theorem) $\left\{ \begin{array}{l} \text{i.e. if } C \text{ is a p-w smooth} \\ \text{closed curve and} \\ p, q \in C'(D \cup C), C = \partial D, \\ \text{then } \oint_C (p dx + q dt) = \int_D (q_x - p_t) dx \end{array} \right.$

$$\text{Similarly, } \int_{D_2} (u\phi_t + f\phi_x) dx dt = \int_{P_1}^{P_2} \phi \cdot (-u_2 dx + f(u_2) dt)$$

$$\Rightarrow \int_{D_1} + \int_{D_2} = \int_{P_1}^{P_2} \phi \cdot ((u_1 - u_2) dx - (f(u_1) - f(u_2)) dt) = 0$$

In view of arbitrariness of ϕ , we must have $(u_1 - u_2) dx - (f(u_1) - f(u_2)) dt = 0$ along Γ .

\Rightarrow since $\frac{dx}{dt} = \dot{s}$ is the shock speed, then

$$\boxed{\dot{s} = \frac{[f]}{[u]}}, \text{ i.e. Rankine-Hugoniot cond's are recovered.}$$

$$[u] = u_2 - u_1$$

Consider

$$u_t + q(u)x = 0$$

$x \in \mathbb{R}$, $t > 0$.

$u(x,t)$ = density or concentration of a physical quantity Q

$$q(u) = \text{flux function} = \frac{[M]}{[T]}$$

$[M]$ = units of mass

$[T]$ = units of time

$$\int_a^b u(x,t) dx = \text{total amount of } Q \text{ in the interval } [a,b].$$

recall

$$\frac{d}{dt} \int_a^b u(x,t) dx = -q(u(b,t)) + q(u(a,t))$$

$\underbrace{\frac{d}{dt} \int_a^b}_{\frac{dQ}{dt}}$ $\underbrace{-q(u(b,t)) + q(u(a,t))}_{\text{net flux at end points}}$

- recall from pollutant problem that, ignoring diffusion,

$$u_t + q u_x = 0 \quad \text{and the constitutive relation for the flux function was}$$

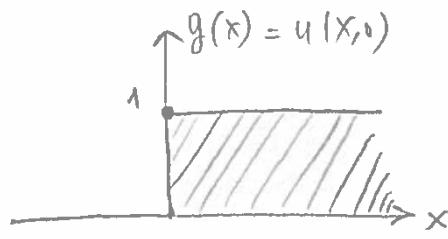
$$q(u) = q u$$

$q \equiv$ what we called U
the "wind" speed.

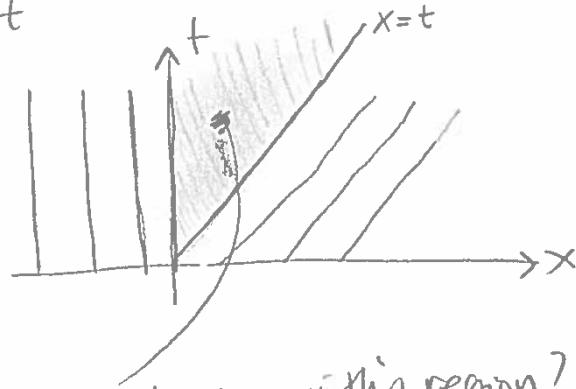
- Example: right propagating model
 $u = \text{density}$

$$u_t + u u_x = 0$$

$$u(x_0) = g(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$



$$\left\{ \begin{array}{l} \frac{dx}{dt} = z \Rightarrow x = x_0 + g(x_0)t \\ \frac{dy}{dt} = 1 \\ \frac{dz}{dt} = 0 \end{array} \right.$$



what happens in this region?
region: $0 < x < t$

Double-checking for quasi-linear case:

$$\text{PDE} = \boxed{F = ap + bq - c = 0}, \quad F_p = a, \quad F_q = b,$$

$$pF_p + qF_q = ap + bq = \boxed{c}. \quad \checkmark$$

The above system $\#$ is NOT Complete.

Need equations for p and q .

Need to understand how p and q "evolve" along charact. curves \Rightarrow "Evolution of a tangent plane".
(plane element)

On a characteristic curve δ :

$$p = \phi(x(\xi), y(\xi)), \quad q = \psi(x(\xi), y(\xi))$$

along δ {

$$\frac{dp}{d\xi} = \phi_x \frac{dx}{d\xi} + \phi_y \frac{dy}{d\xi} \quad \frac{dq}{d\xi} = \psi_x \frac{dx}{d\xi} + \psi_y \frac{dy}{d\xi}.$$

Using 2 first charac. eqns.

~~XX~~ {

$$\begin{aligned} \frac{dp}{d\xi} &= \phi_x F_p + \phi_y F_q \\ \frac{dq}{d\xi} &= \psi_x F_p + \psi_y F_q \end{aligned}$$

- Need to find expressions for $[p_x, p_y, q_x, q_y]$

Use PDE:

$$\frac{dF}{dx} = F_x + p \cdot F_u + p_x F_p + q_x F_q = 0$$

$$\text{Admit } \frac{dF}{dy} = F_y + q \cdot F_u + p_y F_p + q_y F_q = 0$$

Regularity so that

$$p_y = q_x = u_{xy}$$

then

$$\left\{ \begin{array}{l} \text{RHS pag 20} \\ \begin{aligned} p_x F_p + p_y F_q &= -F_x - p F_u \\ q_x F_p + q_y F_q &= -F_y - q F_u \end{aligned} \end{array} \right.$$

The charac. System of ODES is now complete

$$\left\{ \begin{array}{l} \frac{dx}{ds} = F_p \\ \frac{dy}{ds} = F_q \\ \frac{dz}{ds} = p F_p + q F_q \\ \frac{dp}{ds} = -F_x - p F_u \\ \frac{dq}{ds} = -F_y - q F_u \end{array} \right.$$

Lagrange-Charpit eqn

Recap:
these come from ...

$$\frac{\partial H}{\partial p} = -(x-x_0) - \frac{dx}{dp}(y-y_0) = 0$$

$$\left\{ \begin{array}{l} \frac{dF}{dp} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q} \frac{dq}{dp} = 0 \\ \frac{d}{ds} \left(\frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} \right) = 0 \end{array} \right.$$

Law of propagation of characteristic elements
(or of tangent planes)
along charact.

- Cauchy problem : the Cauchy data is

$$x = f(s), \quad y = g(s) \quad \text{e} \quad u = h(s).$$

Contrast with linear and quasi-linear cases;
we now need "initial" values for p and q .
Call it $\phi(s)$ and $\psi(s)$. How to find them?

- ' Let's use what we have up to now :



STRIP CONDITION

$F(f(s), g(s), h(s), p(s), q(s)) = 0$

$(p, q, -1) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) = 0 \quad \textcircled{O}$

$\frac{du(x(s), y(s))}{ds} = \phi(s) \frac{dx}{ds}(s) + \psi(s) \frac{dy}{ds}(s)$

"Initial PDE"
(algebraic cond.)

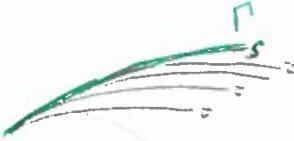
Solving this NONLINEAR set of equations
leads to $\phi(s), \psi(s)$.

if possible.

admissible tangent
planes along Γ
forming a "strip"
(geom. cond.)

- Admitting we CAN solve \square we have a problem with an ODE system of 5 eqns together with 5 "initial" conditions along Γ : $f(s), g(s), h(s), p(s), q(s)$.
Cauchy data

The same condition as before holds:



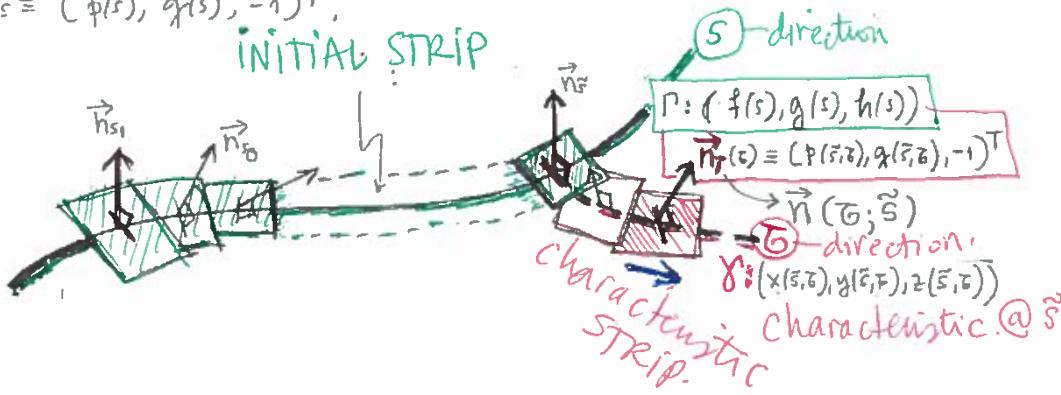
more
specifically

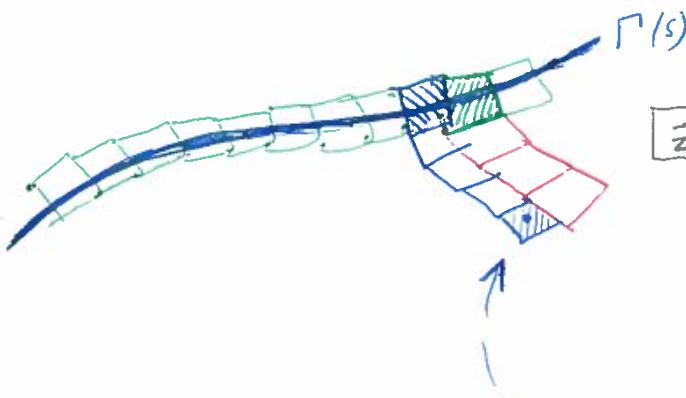
$$\begin{aligned}
 & \quad \downarrow \frac{df}{ds} F_q(f, g, h, p(s), q(s)) - \frac{dg}{ds} F_p(f, g, h, p(s), q(s)) = \\
 & = \begin{vmatrix} f' & g' \\ F_p & F_q \end{vmatrix} \underset{\text{characteristic}}{\longrightarrow} \frac{\frac{df}{ds} \frac{dy}{ds}(s_0) - \frac{dg}{ds} \frac{dx}{ds}(s_0)}{\frac{dy}{ds}(s_0)} \neq 0. \quad \Leftrightarrow \begin{vmatrix} f' & g' \\ a & b \end{vmatrix} \neq 0 \\
 & \quad \text{quasi-lin.} \quad \text{don't have these now}
 \end{aligned}$$

- the initial data is (geometrically speaking) an initial curve Γ with a family of tangent planes (w/ normal vectors $(p, q, -1)$) sliding along Γ .
Hence we form an **INITIAL STRIP:**

$$\vec{n}_s = (\phi(s), \eta(s), -1)^T,$$

INITIAL STRIP





$$z = u(x, y)$$

int. surface can be viewed
as smooth version (locally)
of this "faceted surface"
envelope.

\rightarrow quintuple $= (x, y, z, p, q)^T \equiv$ plane element

Associate a point $(x(\tau), y(\tau), z(\tau))$
on a curve Γ AND a tangent
plane $(\vec{n} = (p, q, -1))$ at
that point.

Note :

$$F(x, y, z, p, q)$$

$$F = \text{constant}$$

is an "integral" of the system.
along any solution slice in the direction of
propagation:

$$\frac{dF}{ds} = F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} + F_p \frac{dp}{ds} + F_q \frac{dq}{ds} = 0$$

↑ ↑ ↑ ↑ ↑
Substitute charac. ODES

family
of
initial
plane
elements

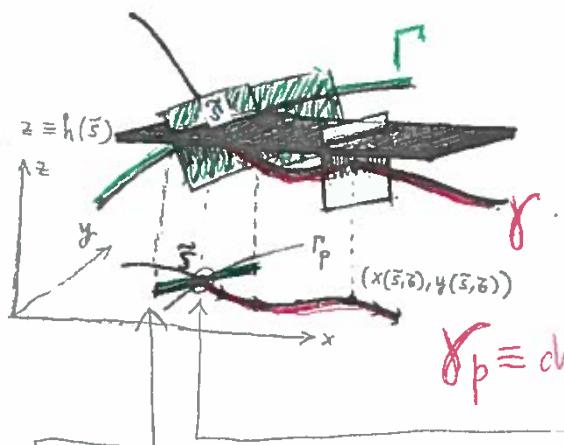
$$(f(\tilde{s}), g(\tilde{s}), h(\tilde{s}), p(\tilde{s}), q(\tilde{s}))^T,$$

$$z - h(\tilde{s}) = p(\tilde{s})(x - f(\tilde{s})) + q(\tilde{s})(y - g(\tilde{s}))$$

Many times it is convenient to look at projections onto the horizontal x-y plane:

Line element

$$0 = \phi(x - \phi(\bar{s})) + g(y - g(\bar{s})) \quad \text{at } (\phi(\bar{s}), g(\bar{s}))$$



$$y = (\phi(\bar{s}) - x) \frac{\phi'(\bar{s})}{g'(\bar{s})} + g(\bar{s})$$

$$\dot{y} = -\left(\frac{\phi'(\bar{s})}{g'(\bar{s})}\right)x + g(\bar{s}) + \frac{\phi(\bar{s})\phi'(\bar{s})}{g'(\bar{s})}$$

γ_p = characteristic projection = RAY

- Line element can sometimes be seen as a wave front (or a phase) locally following a ray,

Back to example
(page 17)

$$\begin{aligned} u_t + u_x &= 1 \\ u(x, 0) &= x \end{aligned}$$

more info

where (IIP)

$$\begin{cases} x = s = f(s) \\ t = 0 = g(s) \\ u = s = h(s) \end{cases}$$

$$\left\{ \begin{array}{l} \frac{dx}{ds} = q \\ \frac{dt}{ds} = p \\ \frac{dz}{ds} = pq + qp - 2pq \\ \frac{dp}{ds} = -f_x - p f_u = 0 \\ \frac{dq}{ds} = -F_y - q F_u = 0 \end{array} \right.$$

$x(s, 0) = s$
 $t(s, 0) = 0$
 $z(s, 0) = s$

$p(s, 0) = ?$ need to find
 $q(s, 0) = ?$

↓ so we use ...

STRIP
CONDITION

$$F(f(s), g(s), h(s), p(s), q(s)) = 0 \Rightarrow \boxed{p(s)q(s) = 1}$$

$\frac{du}{ds} = p \frac{dx}{ds} + q \frac{dt}{ds} \Rightarrow 1 = p \cdot 1 + q \cdot 0 \Rightarrow \boxed{p(s) = 1}$

data compatibility
reduces ambiguity $\Rightarrow \Rightarrow \boxed{q(s) = 1}$

- Solving the ODE system:

$$1st + last \Rightarrow x(s, t) = x_0(s) + q(s)t = s + t$$

$$2nd + 4th \Rightarrow t(s, t) = t(s, 0) + p(s)t = 0 + t$$

$$z(s, t) = z(s, 0) + 2p(s)q(s)t = s + 2t$$

$$z = s + 2t$$

$$u(x, t) = z(s(x, t), t(x, t)),$$

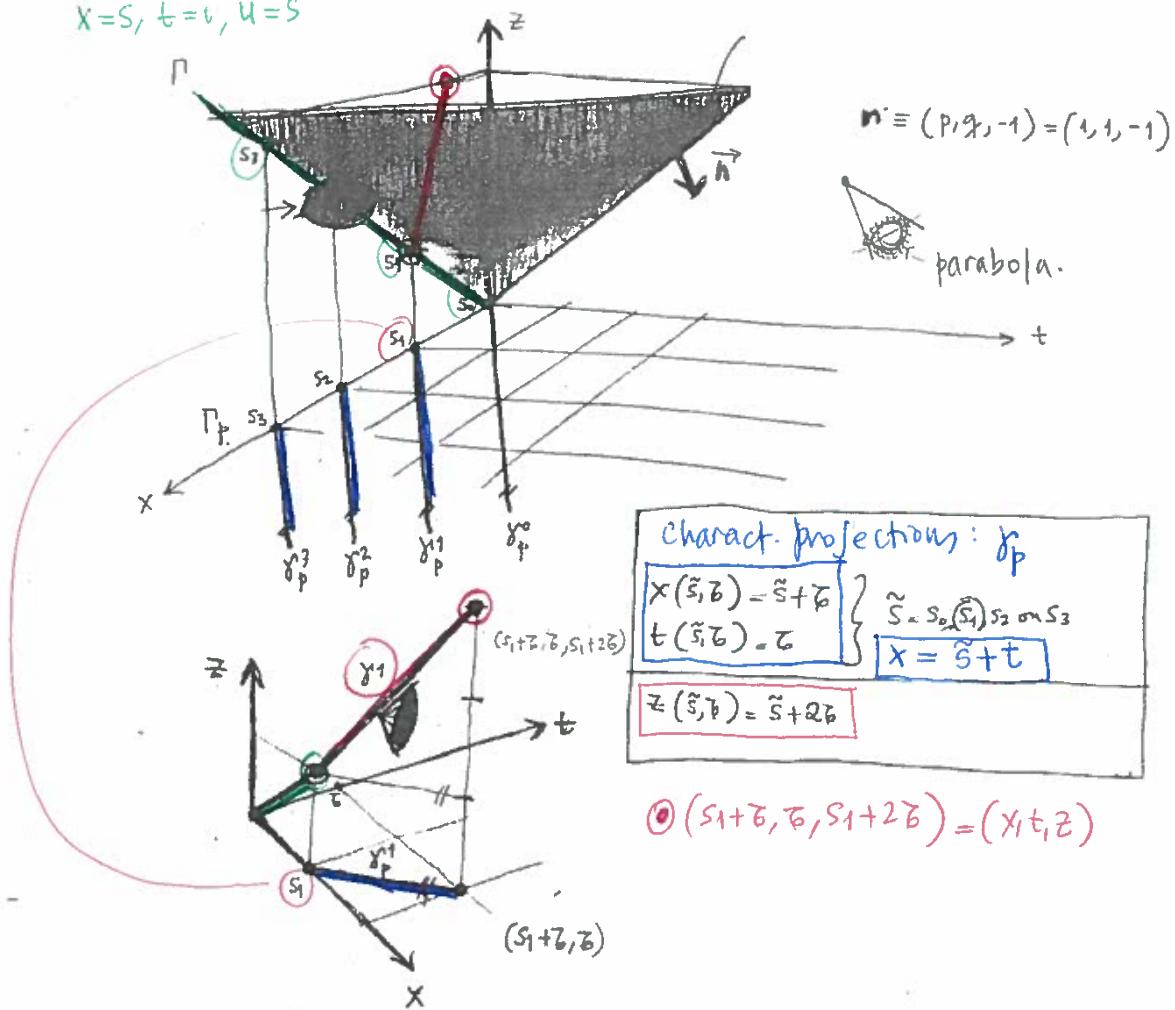
$$\begin{cases} x = s + t \\ t = t \end{cases}$$

$$\begin{cases} "invert" \\ \rightarrow \end{cases} \begin{cases} s = x - t \\ t = t \end{cases}$$

$$u(x, t) = (x - t) + 2t = x + t$$

$$u(x, t) = x + t$$

$$x = s, t = v, u = s$$



① A nonlinear wave-like equation: $c = c(u_x) = u_x$

ITP: $F(x, t, u, \dot{u}, \ddot{u}) = u_t + u_x^2 = 0$, $u(x, 0) = ax$, $a = \text{constant}$
 $\boxed{F = \dot{u} + \dot{u}^2}$ let $a > 0$

$u_t + c u_x = 0$, $c = c(u_x)$.

$c = c(u) = u$.

② Parametrize:

$x(0, s) = s$, $t(0, s) = 0$, $z(0, s) = as$

③ Additional data for p and q :

$$\begin{cases} q(s) + p^2(s) = 0 \\ \frac{du}{ds} = p \frac{dx}{ds} + q \frac{dt}{ds} \Rightarrow a = p^1 + q^0 \end{cases}$$

$\boxed{p(s) = a, q(s) = -a^2}$

④ Charact. system:

$$\left\{ \begin{array}{l} \frac{dx}{d\zeta} = 2p \\ \frac{dt}{d\zeta} = 1 \\ \frac{dz}{d\zeta} = 2p^2 + q \\ \frac{dp}{d\zeta} = 0 \\ \frac{dq}{d\zeta} = 0 \end{array} \right. \Rightarrow \begin{aligned} x(s, \zeta) &= x(s, 0) + 2p(s)\zeta \Rightarrow \boxed{x(s, \zeta) = s + 2a\zeta} \\ t(s, \zeta) &= t(s, 0) + \zeta \Rightarrow \boxed{t(s, \zeta) = 0 + \zeta} \\ z(s, \zeta) &= z(s, 0) + (2p^2 + q)\zeta \Rightarrow \boxed{z(s, \zeta) = as + a^2\zeta} \end{aligned}$$

- therefore

$$\boxed{u(x,t) = z(s(x,t), \bar{s}(x,t))} = as + a^2 t$$

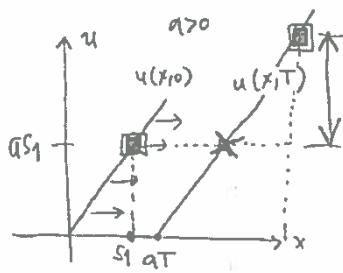
where $x = s + 2at$
 $t = \bar{s}$

$$\Rightarrow \boxed{\begin{aligned} s &= x - 2at \\ \bar{s} &= t \end{aligned}}$$

$$u(x,t) = a(x-2at) + a^2 t = ax - a^2 t = a(x-at)$$

$$\boxed{u(x,t) = a(x-at)}$$

initial condition sets amplitude and speed.



$$C = ux = a > 0$$

$$\begin{aligned} x(s_1, 0) &= s_1 + 0 = \boxed{s_1} \Rightarrow u = ax = \boxed{as_1} \\ x(s_1, T) &= \boxed{s_1 + 2aT} \Rightarrow u = \boxed{as_1} + \boxed{a^2 T} \end{aligned}$$

different "height"
on graph. \uparrow

$$\left\{ \begin{array}{l} u = ax - a^2 t \\ at + t = T, u = 0 \text{ at } x = aT \end{array} \right.$$

- Charact. eqns. in terms of time t.

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 2a \\ \frac{dz}{dt} = 2a^2 + q \Rightarrow \boxed{u \neq \text{const}} \\ \frac{dp}{dt} = 0 \Rightarrow p = \text{const} = u_x \Rightarrow \boxed{\frac{dx}{dt} = 2u_x = \text{constant}} \\ \frac{dq}{dt} = 0 \end{array} \right.$$

- $u(x,t) = a(x-at)$ "wave" profile moves with speed a
- charact. projections: points on these travel with speed $2a$.
- \blacksquare : is moving "up" the wave profile.
- problem if $u_x = \text{changes sign}$.

— 2D Wave equation

$$\nabla_{xx} + \nabla_{yy} = \frac{1}{c^2} \nabla t \quad , \quad c(x,y) = \text{wave speed in heterogeneous medium}$$

time harmonic wave

$$v(x,y,t) = V(x,y) e^{-i\omega t}$$

sep. of variables

$$v = V(x,y) F(t)$$

— Helmholtz equation \longleftrightarrow Reduced wave eq.

$$\nabla_{xx} + \nabla_{yy} + k^2 n^2 V = 0$$

$k_j = \frac{2\pi}{\lambda} = \text{wave number}$

$$\omega = \frac{2\pi}{T}$$

$$\text{Index of refraction } n(x,y) = \frac{\omega}{k c(x,y)} = \frac{c_0}{c(x,y)}$$

$$c_0 = \frac{\omega}{k}, A_0 e^{i(kx - \omega t)} = A_0 e^{i k(x - \frac{\omega}{k} t)}$$

reference speed, k = reference wave number

$\omega = \omega(k)$ = angular frequency (in time)

$$k = \sqrt{k_1^2 + k_2^2} \quad \begin{cases} k_1 = \text{"spatial freq" in x} \\ k_2 = \text{"spatial freq" in y} \end{cases}$$

$k \gg 1$ — recognise — short waves

$$\text{mean } \Rightarrow V(x,y) = A(x,y) e^{ik_2 \ell(x,y)} \quad \begin{matrix} \text{phase} \\ \text{takes care of wave number variation due to heterogeneity} \end{matrix}$$



propag. speed changing

Subst. in Helmholtz eqn. to get

$$A(-2\ell_x^2 - 2\ell_y^2 + n^2) + \frac{i}{k} (2A_x \ell_x + 2A_y \ell_y + A(2\ell_{xx} + 2\ell_{yy})) + \frac{1}{k^2} (A_{xx} + A_{yy}) = 0$$

Since $k \gg 1$ the leading order term gives as a first approximation the eiconal equation

$$\ell_x^2 + \ell_y^2 = n^2$$

for the phase.

For $n = \text{constant}$, $A = \text{constant}$

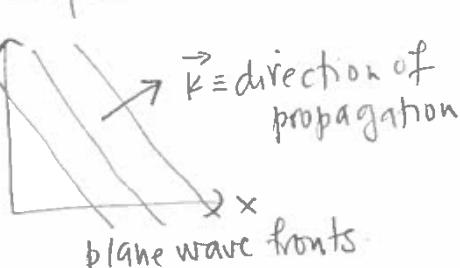
$$\nabla = A e^{i k n (x \cos \theta + y \sin \theta)}$$

is a solution.

Phase $\ell(x, y) = n \frac{\vec{k} \cdot \vec{x}}{k} \Rightarrow$

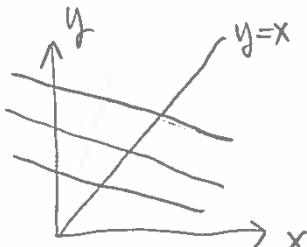
and

$$v(x, y, t) = A e^{i(n \vec{k} \cdot \vec{x} - \omega t)}$$



① Solve $2px^2 + 2qy^2 = n^2$, $n = \text{constant}$

$$2p|_n = ay, \Gamma: x=y$$



projected
view, to be confirmed

— Parametrize Canclng data:

$$x=s, y=s, 2p=as.$$

— Data for p and q :

$$\begin{aligned} \text{strip condition } & \left\{ \begin{array}{l} F = p^2 + q^2 - n^2 = 0 \Rightarrow (p/n)^2 + (q/n)^2 = 1 \\ \frac{dp}{ds} = p \frac{dx}{ds} + q \frac{dy}{ds} \Rightarrow a = p \cdot 1 + q \cdot 1 \end{array} \right. \end{aligned} \quad \begin{array}{c} \oplus_1 \\ \oplus_2 \end{array}$$

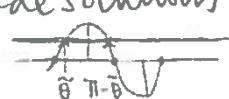
suggests re-writing as

$$\oplus_1 \Rightarrow \boxed{p(s) = n \cos \theta}, \boxed{q(s) = n \sin \theta}$$

$$\oplus_2 \rightarrow \oplus_3 \quad \cos \theta + \sin \theta = \frac{a}{n}$$

$$\text{or } \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right) = \frac{a}{n}$$

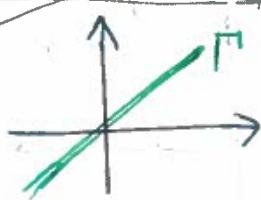
has 2 real solutions if $a \leq \sqrt{2}n$



$$\text{Ex: if } \frac{a}{n} = 1 \quad \theta = 0 \quad \theta = \pi/2,$$

— Condition for non-charac. Γ :

$$\left(\frac{dx}{ds} \right)_q F_q - \left(\frac{dy}{ds} \right)_p F_p \neq 0 \Rightarrow 2q - 2p \neq 0 \Rightarrow \boxed{2n(\cos \theta - \sin \theta) \neq 0} \quad \text{X}$$



$$\begin{cases} \theta \neq \frac{\pi}{4}, \frac{5\pi}{4} \\ a \neq \pm n\sqrt{2} \end{cases}$$

Note if $a = \pm n\sqrt{2}$
 $\sin \left(\theta + \frac{\pi}{4} \right) = \pm 1 \Rightarrow \theta = \pi/4, 5\pi/4$

PDE

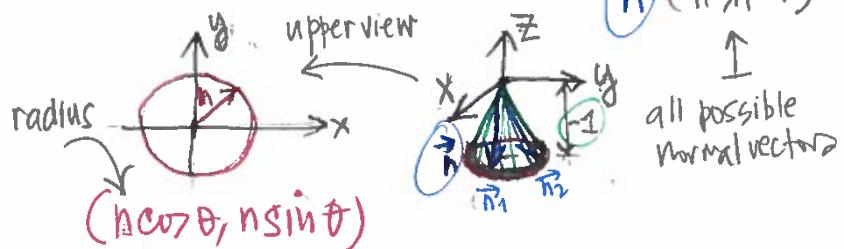
$$\rightarrow \left(\frac{p}{n}\right)^2 + \left(\frac{q}{n}\right)^2 = 1 \quad , \quad n = \text{refraction index}$$

$$\begin{cases} p(s) = n \cos \theta \\ q(s) = n \sin \theta \end{cases}$$

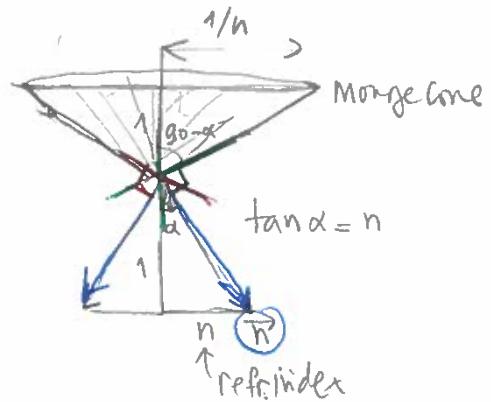
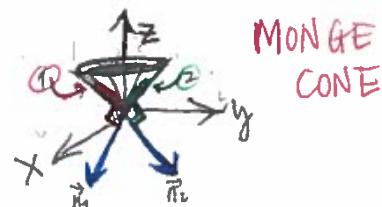
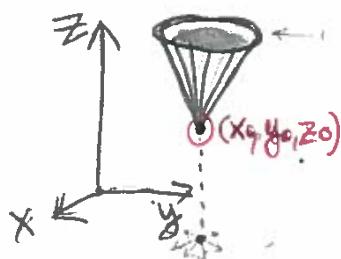
\Rightarrow plane elmts.

1-parameter family in θ

$$(n \cos \theta, n \sin \theta, -1)^T = \text{normal} \cdot \vec{n} = (p, q, -1)^T$$



Eiconal with



④ Charac ODES : $F = p^2 + q^2 - n^2 = 0$

$$\begin{aligned}
 & F_p \quad \left\{ \begin{array}{l} \frac{dx}{ds} = 2p \\ \frac{dy}{ds} = 2q \end{array} \right. \Rightarrow x(s, \theta) = x(s, 0) + 2p s = s + 2n s \cos \theta = x \quad *_1 \\
 & F_q \quad \Rightarrow y(s, \theta) = y(s, 0) + 2q s = s + 2n s \sin \theta = y \quad *_2 \\
 & PFP + qFq \quad \left\{ \begin{array}{l} \frac{dz}{ds} = 2(p^2 + q^2) \\ \frac{dp}{ds} = 0 \\ \frac{dq}{ds} = 0 \end{array} \right. \Rightarrow z(s, \theta) = z(s, 0) + 2n^2 s = (c_1 \theta + s \sin \theta) n s + 2n^2 s \\
 & -F_x - p F_u \quad \\
 & -F_y - q F_u
 \end{aligned}$$

Solve $*_1$ and $*_2$ for

$$s = \frac{y - x}{2n(\sin \theta - \cos \theta)}, \quad z = \frac{x \sin \theta - y \cos \theta}{\sin \theta - \cos \theta}$$

Substitute in $z(s, \theta; \theta) = n s (\cos \theta + \sin \theta) + 2n^2 s$ we get

$$\varphi(x, y) = n(x \cos \theta + y \sin \theta) \rightarrow \text{satisfies Cauchy data} \\ \rightarrow \text{satisfies eiconal eq.}$$

- Slightly more general plane wave satisfying eiconal eq.

$$\varphi(x, y) = \varphi_0 + n((x - x_0) \cos \theta + (y - y_0) \sin \theta), \quad x_0, y_0, \varphi_0 \text{ const}$$

but different Cauchy data.

$$q = n = 1$$

$$\varphi = \alpha x$$

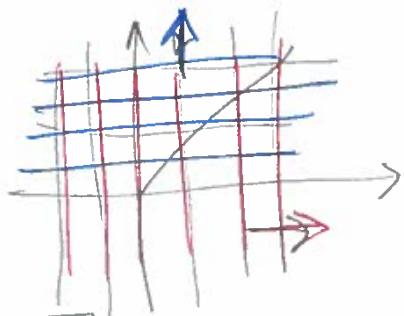
33A

Q7

$$\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) = 1 = \frac{a}{n}$$



$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \theta = 0, \text{ or } \theta = \frac{\pi}{2}$$



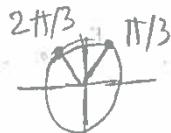
$$a = \sqrt{\frac{3}{2}} n$$

$$\vec{k}_0 = (\cos 0, \sin 0) = (1, 0)$$

$$\vec{k}_{\frac{\pi}{4}} = (0, 1)$$

Q8

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \theta = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{2} \quad (60^\circ)$$



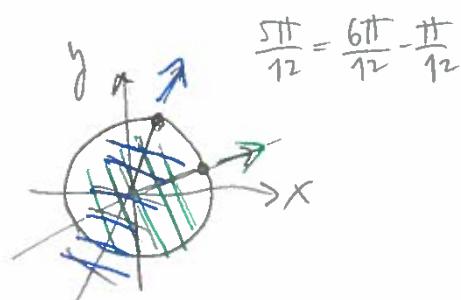
$$\theta + \frac{\pi}{4} = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

$$\theta = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{8\pi - 3\pi}{12} = \frac{5\pi}{12}$$

$$\theta = \pm \frac{\pi}{12}$$

$$\vec{k}_1 = \cos \pi/12, -\sin \pi/12$$

$$\vec{k}_2 = \cos \pi/12, \sin \pi/12$$



Q9

$$\begin{cases} x = s + 2n\bar{v} \cos \theta \\ y = s + 2n\bar{v} \sin \theta \end{cases}$$

$$x+y = s + 2n\bar{v}(\cos \theta + \sin \theta)$$

$$y-x = 2n\bar{v}(\sin \theta - \cos \theta)$$

$$\boxed{\frac{y-x}{2n(\sin \theta - \cos \theta)} = \bar{v}}$$

$$x+y = s + \frac{(y-x)}{\sin \theta - \cos \theta} (\sin \theta + \cos \theta)$$

$$(x+y)(\sin \theta + \cos \theta) = s + (y-x)(\sin \theta + \cos \theta)$$

$$\cancel{x(\sin \theta + \cos \theta) - y(\sin \theta + \cos \theta)} = s + \cancel{y(\sin \theta + \cos \theta)} + \cancel{-x(\sin \theta + \cos \theta)}$$

$$\cancel{x(\sin \theta + \cos \theta) - y(\sin \theta + \cos \theta)} = s$$

$$\boxed{s = -\frac{x \sin \theta - y \cos \theta}{\sin \theta - \cos \theta}}$$

$$\mathcal{L}(x,y) = n(x \cos \theta + y \sin \theta)$$

on Γ : $x=y$, $\mathcal{L} = nx(\cos \theta + \sin \theta) = nx \frac{a}{n} = ax$

$$\begin{aligned} \mathcal{L}_x &= n \cos \theta \\ \mathcal{L}_y &= n \sin \theta \end{aligned} \quad \left\{ \begin{array}{l} \mathcal{L}_x^2 + \mathcal{L}_y^2 = n^2 \\ \end{array} \right.$$

① Solve $\nabla \varphi^2 + \nabla \psi^2 = n^2$

with the following (singular) data at a point:

$$x(s) = x_0, y(s) = y_0, \psi(s) = \psi_0.$$

STRIP CONDITION

$$\frac{d\psi}{ds} = p \frac{dx}{ds} + q \frac{dy}{ds}$$

$$F(x_0, y_0, \psi_0, p(s), q(s)) = 0$$

$$p^2(s) + q^2(s) = n^2 \Rightarrow \begin{cases} p(s) = n \cos(s) \\ q(s) = n \sin(s) \end{cases}, s = \text{const.}$$

— Same charac. ODEs as before

$$\begin{aligned} x(s, \bar{s}) &= x_0 + 2n \bar{s} \cos(s) \\ y(s, \bar{s}) &= y_0 + 2n \bar{s} \sin(s) \\ z(s, \bar{s}) &= \psi_0 + 2n^2 \bar{s} \end{aligned} \Rightarrow \begin{aligned} (x-x_0)^2 &= 4n^2 \bar{s}^2 \cos^2(s) \\ (y-y_0)^2 &= 4n^2 \bar{s}^2 \sin^2(s) \\ \bar{s}^2 &= \frac{1}{4n^2} [(x-x_0)^2 + (y-y_0)^2] \end{aligned} \quad (+)$$

$$\bar{s} = \frac{1}{2n} \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Finally

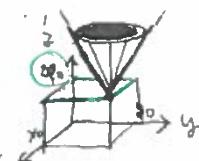
$$\psi(x, y) = z(s, \bar{s}) = \psi_0 + n \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

singular solution
ux, uy blow-up at (x_0, y_0) .

$$(q - \psi_0)^2 = n^2 ((x-x_0)^2 + (y-y_0)^2)$$

$$\varphi = \varphi_1 > \varphi_0 \quad (x-x_0)^2 + (y-y_0)^2 = \left(\frac{\varphi_1 - \varphi_0}{n}\right)^2,$$

larger index of refraction $\Rightarrow c < c_0 \Rightarrow$ slower waves in medium.



② Hyperbolic Systems

not so simple when with variable coefficients

- Examples of some simple linear 2×2 systems:

$$\begin{cases} \rho u_t + p_x = 0 \\ \frac{1}{\rho} p_t + u_x = 0 \end{cases}$$

similar

$$\begin{cases} u_t + g \eta_x = 0 \\ \eta_t + (h(x)u)_x = 0 \end{cases}$$

Acoustic waves, $\rho(z)$ = density
 u = veloc., p = pressure

$\frac{1}{\rho}(z)$ = compressibility

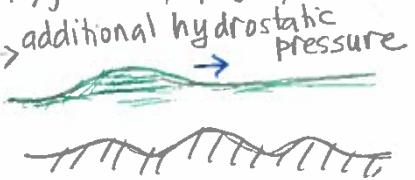
$c(z) = \sqrt{\frac{1}{\rho}(z)}$ = propag. speed

Shallow water model

u = veloc. η = wave elevation

use $\eta_t(x,t) = h(x)u(x,t)$

$$c(x) = \sqrt{gh(x)}$$



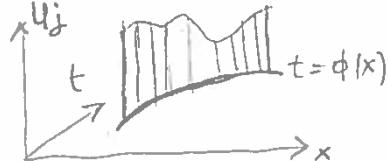
Consider

$$A(x,t) \vec{u}_t + B(x,t) \vec{u}_x = C(x,t) \vec{u} + D(x,t)$$

where $\vec{u} = [u_1, \dots, u_N]^T$, $A, B, C \in N \times N$ matrices, D = vector

Cauchy problem has data given along curve $t = \phi(x)$, namely

$$\vec{u}(x, \phi(x)) = \vec{f}(x)$$



We say the initial curve is charact. if we cannot compute derivatives of u_j from the Cauchy data.
 (for a similar case, check back at page 9)

"initial" PDE
 analog of the strip condition
 (but this problem is linear - easy to solve)

$$\left\{ \begin{array}{l} A \vec{u}_t + B \vec{u}_x = C \vec{u} + D \\ \vec{u}_x + \phi'(x) \vec{u}_t = \vec{f}' \Rightarrow \vec{u}_x = \vec{f}' - \phi'(x) \vec{u}_t \end{array} \right.$$



Substituting \vec{u}_x in the "initial" PDE we get.

$$(A - \phi' B) \vec{u}_t = C \vec{f} + D - B \vec{f}' \quad (\text{ODE-like})$$

We can not compute \vec{u}_t when

$$\det(A - \phi' B) = 0 \Rightarrow \phi(x) \text{ is charac.}$$

or

$$\det(A dx - Bd\phi) = 0.$$

And can not get \vec{u}_x from pag 35.

- Suppose that $\det A \neq 0$ so that we can write the syst. in the form

$$\vec{u}_t + \tilde{B} \vec{u}_x = \tilde{C} \vec{u} + \tilde{D}, \quad \tilde{B}, \tilde{C}, \tilde{D} \text{ - new matrices due to } A^{-1}$$

- Drop the $\tilde{\cdot}$
- The characteristic equations are

$$\det\left(\frac{dx}{dt} I - B\right) = 0$$

or

$$\frac{dx}{dt} = \lambda_i(x, t)$$

The system being hyperbolic implies that the

e-values of B (or $A^{-1}B$) are real $\Rightarrow N$ propagation speeds.

And e-vectors are linearly indep. (LI)

$\Leftrightarrow B$ is diagonalizable

e-vector k , linearly independent

$$B \vec{s}^k = \lambda_k \vec{s}^k,$$

e-value k , $k=1, 2, \dots, N$, real

Λ = diagonal with λ_k^i

Γ = modal matrix w/
e-vectors (columns)

— Make the change of variables

$$B\Gamma = \Lambda\Gamma, \quad \Lambda \text{ is diagonal}$$

$$\vec{v} = \Gamma^{-1} \vec{u}$$

to get

$$\vec{\dot{v}}_t + \Lambda \vec{v}_x = \tilde{C}\vec{v} + \tilde{D}$$

$$\Gamma^{-1} B \Gamma = \Lambda$$

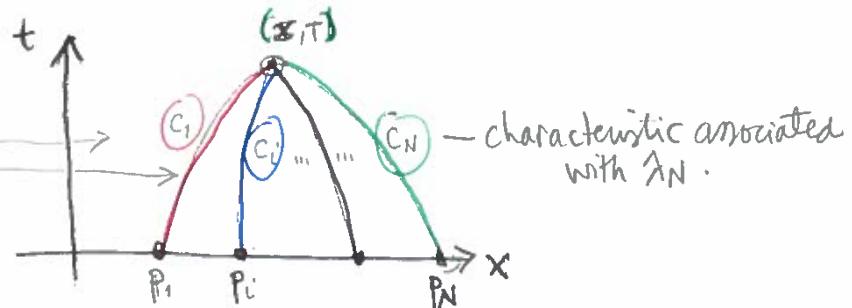
$$\begin{cases} \tilde{C} = \Gamma^{-1} C \Gamma - \Gamma^{-1} \Gamma_t - \Gamma^{-1} B \Gamma_x \\ \tilde{D} = \Gamma^{-1} D \end{cases}$$

accordingly

$$\vec{v}(x_0) = \Gamma^{-1} \vec{f}(x) \equiv \vec{g}(x)$$

canonical form of the system

→ The "picture" is a bit like this:



- Note:
 - Classical (2nd order) wave eq \Rightarrow 2 charact.
 - System $N \times N \Rightarrow N$ charact.
 - Flow of "information" is more complicated.

- Let's look by component:

$$\frac{d\alpha_i}{dt} = \frac{\partial \alpha_i}{\partial t} + \frac{dx}{dt} \frac{\partial \alpha_i}{\partial x} = \sum_k \tilde{c}_{ik} v_k + \tilde{d}_i$$

↑ coupling term

Comment: if $RHS=0$

form of trajectory $\frac{dx}{dt} = \dot{x}_i(x_i, t)$

Comment: given $\rightarrow (x_i, t)$, one can find

$x = \alpha_i(t; x_i, t) \Rightarrow \alpha_i$'s as the departure points: D

(x_i, t) arrival points: A

When $RHS=0$, bring the value along c_i . (x)

integrating along c_i

$v_i(x_i, t) = v_i(\alpha_i(0; x_i, t), 0) + \int_0^T (\sum_k \tilde{c}_{ik} v_k + \tilde{d}_i) dt$

$v_i(\alpha_i(0; x_i, t), 0) = g_i(\alpha_i(0; x_i, t))$

(*) semi-Lagrangian, Eulerian-Lagrangian operate a bit like this.
 Beware: Lagrangian trajectory \neq characteristics in general

— Comment:

Stansforth & Côté – Semi-Lagrangian Integration Scheme for Atmospheric Models – A Review

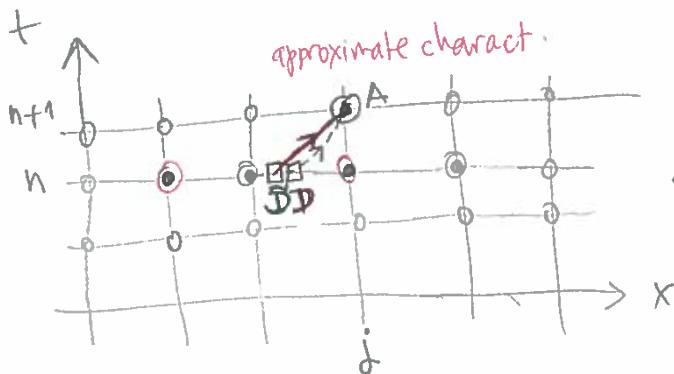
Monthly Weather Review, '91.

— Examples in passive^{*}-advection

$$\textcircled{1} \quad F_t + \vec{U}(x, t) \cdot \nabla F = 0 \quad \rightarrow \quad \frac{dx}{dt} = \vec{U}(x, t)$$

$$\textcircled{2} \quad F_t + \vec{V}(x, t) \cdot \nabla F = R(x, t) \quad \rightarrow \quad \frac{d\vec{x}}{dt} = \vec{V}(x, t)$$

charac = Lagrangian formulation



in this case the Eulerian-Lagrangian method is schematically like this.

$$\textcircled{1} : \frac{dF(x(t; x_D))}{dt} = 0 \quad \begin{matrix} D = \text{departure pt.}, \tilde{D} = \text{approximate depart. pt.} \\ A = \text{arrival pt.} \end{matrix}$$

$$F(x(t^{n+1}; x_A)) = F(x(t^n; x_D)) \approx F(x(t^n; x_{\tilde{D}}))$$

$F(x(t^n; x_{\tilde{D}}))$ = cubic interpolation at level $t=t^n$ \bullet
 \tilde{D}, \tilde{D} = not grid point, in general

* in opposition to, for example,

$$\frac{Du}{Dt} = u_t + u u_x$$

transport depends on solution or $C(u)$

- Back to our acoustic and shallow water models:
(long waves)

- We saw that



Acoustic waves

$$\begin{cases} \rho u_t + p_z = 0 \\ \frac{1}{\rho} p_t + q_z = 0 \end{cases}$$

where

$$\begin{cases} \rho = \rho(z), \\ \frac{1}{\rho} = h^{-1}(z), \\ c(z) = \sqrt{\frac{h'(z)}{\rho(z)}} \end{cases}$$

- Write system as

$$\begin{cases} u_t + \frac{1}{\rho} p_z = 0 \\ p_t + \frac{1}{\rho} u_z = 0 \end{cases}$$

$$\begin{bmatrix} u \\ p \end{bmatrix}_t + \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\rho} & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}_z = 0$$

$$\begin{vmatrix} -\lambda & \frac{1}{\rho} \\ \frac{1}{\rho} & -\lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda^2 - \frac{1}{\rho} = 0}$$



Shallow water waves

$$\begin{cases} \eta_t + g \eta_x = 0 \\ \eta_t + (h(x)u)_x = 0 \end{cases}$$

$\eta = \eta(x, t)$ — wave elevation

$h(x)$ = water depth



$$(f(x) - \psi(x, t)) \equiv h(x) u(x, t)$$

$$\begin{bmatrix} \psi \\ \eta \end{bmatrix}_t + \begin{bmatrix} 0 & gh \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \eta \end{bmatrix}_x = 0 \Leftrightarrow \vec{U}_t + B_1(x) \vec{U}_x = 0$$

$$\lambda^2 - gh(x) = 0$$

$$\lambda = \pm c_0(x)$$

$$c_0(x) = \sqrt{gh(x)}$$

called the shallow water speed.

not shallow water but long wave: $a \approx 1\text{m}$, $\lambda \approx 50-100\text{ km}$

$$\text{tsunami} \approx \sqrt{10\text{ m} \cdot 4000\text{ m}} = 200\frac{\text{m}}{\text{s}}$$

$$c_0 = 720 \text{ kmph} \approx 450 \text{ mph}$$

$$\text{as we will see "shallow water parameter"} = \left(\frac{h}{\lambda}\right)^2 = \left(\frac{4}{100}\right)^2 = 0.0016$$

① A nice trick. Let's use it with acoustic waves.

Define travel time x as

$$x(z): \quad x = \int_0^z \frac{1}{c(s)} ds$$

Sometimes also denoted by $G(z)$
(time it takes to travel z -length units)

check units

$$\int \frac{[\text{T}]}{[\text{L}]} d[\text{L}] = [\text{T}]$$

check constant speed

$$\int_0^{N\text{-miles}} \frac{1}{60\text{mph}} dx = \frac{N\text{-miles}}{60\text{mph}} = \frac{N}{60} \text{ hours.}$$

- Put the acoustic system in these two time-variables:
 $t = \text{"clock time"}$ $x = \text{travel time}$

$$\begin{cases} S(x) u_t + p_x = 0 \\ p_t + S(x) u_x = 0 \end{cases}$$

$$\begin{array}{c} \left[\begin{matrix} 0 & 1/S \\ S & 0 \end{matrix} \right] \text{ Coupled} \\ \downarrow \\ \vec{U}_t + B_2(x) \vec{U}_x = 0 \\ \lambda = \pm 1 \end{array}$$

$$\text{Impedance} \equiv S(x) = \sqrt{p(x)k(x)}$$

Now make a change of dependent variables indicated below at (*) to get

$$\begin{cases} A_t + A_x = -r(x) B \\ B_t - B_x = r(x) A \end{cases}$$

$$\begin{array}{c} \left[\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right] \text{ decoupled} \\ \downarrow \\ \vec{U}_t + B_3 \vec{U}_x = D \\ \lambda = \pm 1 \end{array}$$

$$(*) \quad A(x,t) \equiv S^{1/2} \bar{u} + \bar{S}^{-1/2} \bar{p}, \quad B(x,t) \equiv -S^{1/2} \bar{u} + \bar{S}^{-1/2} \bar{p}$$

reflectivity \rightarrow

$$r(x) \equiv \frac{S'(x)}{2S(x)} = \frac{1}{2} \frac{d}{dx} \log(S(x))$$

Comments:

(1) Charact. are straight & $\frac{dx}{dt} = \pm 1$ normalized

(2) Homogeneous medium $r(x) \equiv 0$ (A and B decouple)

A, B are right and left
TRAVELLING MODES

- When $\mathcal{S}'(x) \equiv 0$, we have no coupling between modes.

$$\left\{ \begin{array}{l} \frac{dA}{dt}(x(t), t) = 0, \text{ along } \frac{dx^+}{dt} = +1 \\ \frac{dB}{dt}(x(t), t) = 0, \text{ along } \frac{dx^-}{dt} = -1 \end{array} \right.$$

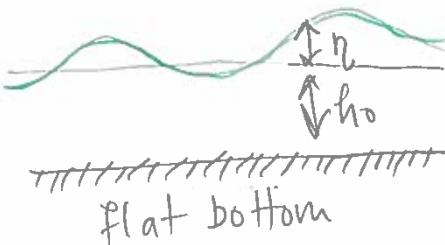
- In 2×2 systems we can find these objects that are constant (invariant) along charact. They are called Riemann Invariants. We can find them in nonlinear hyperbolic systems*. And there are problems where they are not invariant (they are coupled as above) but they are still useful and, abusing of terminology, we call them Riemann Invariants.

Let's see a trick for a nonlinear case:

○ Shallow water (long wave) model.

$$\left\{ \begin{array}{l} u_t + uu_x + gHx = 0 \\ H_t + (uH)_x = 0 \end{array} \right.$$

total water height. $\{ H(x,t) \equiv \eta(x,t) + h_0 \}$



*They are specific to 2×2 systems.
Not for $N > 2$.

- In conservation form: $\begin{bmatrix} u \\ H \end{bmatrix}_t + \begin{bmatrix} u^2/2 + gH \\ -uH \end{bmatrix}_x = 0$ 43

- In non-conservative form, to see matrix and e-values,

$$\begin{bmatrix} u \\ H \end{bmatrix}_t + \begin{bmatrix} u & g \\ H & u \end{bmatrix} \begin{bmatrix} u \\ H \end{bmatrix}_x = 0$$

The system is strictly hyperbolic \Rightarrow 2 distinct real e-values

$$(A-u)^2 - gH = 0$$

$$\lambda(x) = u \pm \sqrt{gH} = u \pm c(x)$$

$$H = \eta + h_0$$

$$= c(x)(F \pm 1)$$

$$Frude \equiv \frac{u}{\sqrt{gH}}$$

both charact
can be forward.

$Fr < 1$ subcritical
 $Fr > 1$ supercritical

- For convenience write the above system in terms of

$$c(x) = \sqrt{gH(x)}$$

$$dc(x) = g \frac{dH(x)}{2c(x)}.$$

- The SW-eqs can be written as

$$\begin{cases} u_t + uu_x + 2cc_x = 0 \\ 2ct + cu_x + 2uc_x = 0 \end{cases}$$

- Add and subtract:

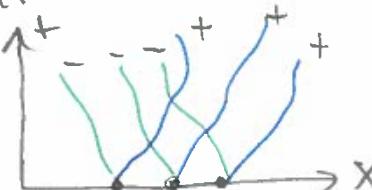
$$\begin{cases} \left(\frac{\partial}{\partial t} + (u+c) \frac{\partial}{\partial x} \right) (u+2c) = 0 \\ \left(\frac{\partial}{\partial t} - (u-c) \frac{\partial}{\partial x} \right) (u-2c) = 0 \end{cases}$$

Let:

$$\begin{cases} \frac{dx^+}{dt} = u+c \\ \frac{dx^-}{dt} = u-c \end{cases} \text{ and}$$

$$A = u+2c, B = u-2c$$

$$\begin{cases} \frac{dA}{dt}(x^+(t), t) = 0 \\ dB/dt(x^-(t), t) = 0 \end{cases}$$



① Linear constant coeff. = easy

43A

$$\vec{u}_t + A \vec{u}_x = 0$$

$$A = M - \Lambda M^{-1}, \quad \Lambda = \text{diag.}$$

$$\text{or } M^{-1}AM = \Lambda$$

M = matrix with right e-values

$$(M^{-1}\vec{u})_t + \cancel{M^{-1}AM} \cancel{M^{-1}} \vec{u}_x = 0$$

$$\vec{w}_t + -\Lambda \vec{w}_x = 0 \quad \Leftrightarrow \quad \frac{dw_i}{dt} = 0, \quad \frac{dx_i}{dt} = \lambda_i$$

easy to decompote

② Quasi-linear $M = M(\vec{u})$

$$M^{-1}\vec{u}_t \neq (M^{-1}\vec{u})_t$$

$$M^{-1}\vec{u}_x \neq (M^{-1}\vec{u})_x.$$

- Consider the 2×2 strictly hyperbolic syst.

$$\vec{u}_t + A(\vec{u}) \vec{u}_x = 0,$$

where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $a_{ij} = a_{ij}(u_1, u_2)$,

- Consider the left \star e-vectors \vec{L}_1, \vec{L}_2 $|A - \lambda I| = 0$.

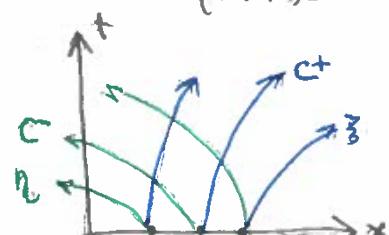
$$\begin{aligned} \vec{L}_i (\vec{u}_t + A \vec{u}_x) &= (\vec{L}_i \vec{u}_t + \lambda_i \vec{L}_i \vec{u}_x) = \\ &= \vec{L}_i \left(\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x} \right) \vec{u} = \vec{L}_i \left(\frac{\partial}{\partial t} + \left(\frac{dx}{dt} \right)_i \frac{\partial}{\partial x} \right) \vec{u} = 0 \end{aligned}$$

- We have that

$$\left\{ \vec{L}_1 \frac{d\vec{u}}{d\xi} = 0, \text{ along } C^+ : \frac{dx}{dt} = \lambda_1 \right. \\ \left. (\xi(x,t) \text{ constant}) \right.$$

$$\left\{ \vec{L}_2 \frac{d\vec{u}}{d\eta} = 0, \text{ along } C^- : \frac{dx}{dt} = \lambda_2 \right. \\ \left. (\eta(x,t) \text{ constant}) \right.$$

$$\left\{ \ell_{11} \frac{du_1}{d\xi} + \ell_{12} \frac{du_2}{d\xi} = 0, \text{ on } C^+ \right. \\ \left. \ell_{21} \frac{du_1}{d\eta} + \ell_{22} \frac{du_2}{d\eta} = 0, \text{ on } C^- \right.$$



$$\star \quad \vec{L} A = \lambda \vec{L} \quad \Rightarrow \quad \square = \lambda \square \\ (\vec{L} A)^T = (\lambda \vec{L})^T \Rightarrow A^T \vec{L}^T = \lambda \vec{L}^T \quad \square \square = \lambda \square \\ \text{right e-vector of transpose.}$$

Integrate these 2ODEs when they are an EXACT DIFFERENTIAL

$$\exists \Psi_1, (\ell_1, \ell_{12}) = \nabla \Psi_1$$

$$\exists \Psi_2, (\ell_2, \ell_{22}) = \nabla \Psi_2$$

2ODEs $\begin{cases} \ell_{11}(u_1, u_2) du_1 + \ell_{12}(u_1, u_2) du_2 = 0, \text{ along } C^+ \\ \ell_{21}(u_1, u_2) du_1 + \ell_{22}(u_1, u_2) du_2 = 0, \text{ along } C^- \end{cases}$

like
this

$$M(x,y) dx + N(x,y) dy = 0$$

or use an integrating factor $\mu = \mu(u_1, u_2)$

to get

\downarrow "constant of integration".

$$\int \mu \ell_{11} du_1 + \int \mu \ell_{12} du_2 = r(\eta), \quad \text{along } C^+,$$

$$\int \tilde{\mu} \ell_{21} du_1 + \int \tilde{\mu} \ell_{22} du_2 = s(\zeta), \quad \text{along } C^-.$$

$r(\eta)$ & $s(\zeta)$ \equiv Riemann invariants

— More details, see Debnath page 329.

① Second order PDEs

Consider the quasi-linear PDE

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

where a, b, c, d are functions of x, y, u, u_x, u_y .

Take the curve $\Gamma_p : (x = f(s), y = g(s))$ where we have the data $u = h(s)$, $u_x = p(s)$ & $u_y = q(s)$. 2nd order - data is given.

From the strip condition $(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) \cdot (p, q, -1) = 0$

$$h'(s) = p(s)f'(s) + q(s)g'(s) \quad \leftarrow \text{no free parameter} \rightarrow \text{quasi-linear}$$

Actually $f(s), g(s)$ and $h(s)$ need to satisfy a compatibility condition:

Cannot be imposed arbitrarily.

The freedom (along Γ_p) is to impose u and $\frac{du}{dn}$ (normal to Γ_p):

$$u = h(s), \quad \frac{du}{dn} = \frac{-u_x g' + u_y f'}{\sqrt{f'^2 + g'^2}} = \chi(s)$$

$$\nabla u \cdot \vec{n}_p, \quad \vec{n}_p = \frac{(-g', f')}{\sqrt{f'^2 + g'^2}}$$

↑
unit normal

- As before compatibility condition arises in obtaining initial values for highest order derivatives from the data:

$$\left\{ \begin{array}{l} \frac{du_x}{ds} = u_{xx}f' + u_{xy}g', \text{ where } \begin{cases} u_x = p(s) \\ u_y = q(s) \end{cases} \\ \frac{du_y}{ds} = u_{yx}f' + u_{yy}g' \end{array} \right.$$

Using the "initial" PDE we have :

$$\left\{ \begin{array}{l} a u_{xx} + 2b u_{xy} + c u_{yy} = d \\ f' u_{xx} + g' u_{xy} = p' \\ f' u_{xy} + g' u_{yy} = q' \end{array} \right. \quad]$$

$$\begin{bmatrix} \dots \\ \vdots \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} d \\ p' \\ q' \end{bmatrix}$$

Compatibility Cond.

$$\Delta = \begin{vmatrix} f' & g' & 0 \\ 0 & f' & g' \\ q & 2b & c \end{vmatrix} = ag'^2 - 2bf'g' + cf'^2 \neq 0$$

Def: $\Delta = 0, \text{ or } \Gamma_p \rightarrow \Gamma_p \text{ is characteristic}$
 $\Delta \neq 0 \Rightarrow \Gamma_p \text{ is NOT charac.}$

- A local power series expansion (solution) can be constructed by repeating the above steps for u_{xxx}, u_{xyx}, \dots etc.. evaluated at a point along Γ_p .
- For example the "initial" PDE

$$\partial_x [a u_{xx} + 2b u_{xy} + c u_{yy} = d] \text{ and so on.}$$
- Convergence of power series \star - analyticity structure of solutions, and so on, can follow from Cauchy-Kowalevski Theorems.
(different version)
 \star Existence of solutions

2m

- Take the compatibility condition (compute determinant)

$$a g^2 - 2bf'g' + cf'^2 = a \left(\frac{dy}{ds}\right)^2 - 2b \frac{dg}{ds} \frac{dx}{ds} + c \left(\frac{dx}{ds}\right)^2.$$

which is a quadratic equation for dy/dx with roots

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

elliptic	$b^2 < ac$	$\rightarrow \nexists \text{ real roots}$
parabolic	$b^2 = ac$	$\rightarrow 1 \text{ root}$
hyperbolic	$b^2 > ac$	$\rightarrow 2 \text{ real roots} - \text{charact.}$

Note: A PDE can change type depending on region
 Ex: Tricomi eq., $u_{yy} - y u_{xx} = 0$

Application = (flow in porous media)
 Conservation Law of mixed type - have elliptic regions, usually associated with instabilities.

- Take linear 2nd order PDES

$$Lu = \boxed{a u_{xx} + 2b u_{xy} + c u_{yy} + 2d u_x + 2e u_y + fu = 0}$$

a, b, c, d, e, f functions of (x, y) .

Let $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ be a new coordinate system.

$$\xi = \phi(x, y), \quad \eta = \psi(x, y),$$

$\xi = \text{constant}$ is a level-curve in this new coord-sys
where only η varies.

In these new coordinates

$$\boxed{Lu = A(\xi, \eta) u_{\xi\xi} + 2B(\xi, \eta) u_{\xi\eta} + C(\xi, \eta) u_{\eta\eta} + \dots = 0} \quad \checkmark$$

$$\text{where } A = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2$$

$$B = a\phi_x^2\psi_x + b(\phi_x^2\psi_y + \phi_y^2\psi_x) + c\phi_y^2\psi_x$$

$$C = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2$$

Note! If ϕ and ψ represent characteristic curves
then $A = C = 0$. If $B \neq 0$ we divide through to

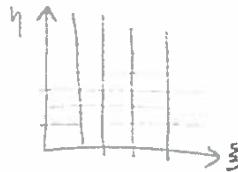
get

$$\boxed{u_{\xi\xi} + 2D u_\xi + 2E u_\eta + Fu = 0}$$

The equation above has characteristics

$$\xi = \text{constant} \quad \& \quad \eta = \text{constant}$$

and if we use



$$\begin{cases} x' = \xi + \eta \\ y' = \xi - \eta \end{cases}$$

we get a wave eq. with normalized speed:

$$\boxed{4yy' - 4x'x'' + 2D'u_{x'} + 2E'u_{y'} + F'u = 0}.$$



in the elliptic case: can we reduce to an equation similar to Laplace's equation? Namely starting from □1, page 48 and getting

$$\boxed{4z_z + 4\eta\eta + 2D'u_z + 2E'u_\eta + F'u = 0}.$$

In this case we need to impose $A = C$ and $B = 0$. This will lead to the equations

$$\phi_x = \frac{b\bar{\psi}_x + c\bar{\psi}_y}{w}, \quad \phi_y = -\frac{a\bar{\psi}_x + b\bar{\psi}_y}{w}$$

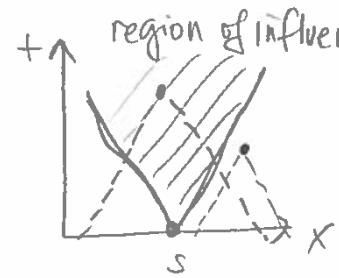
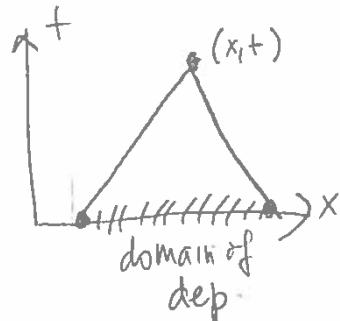
$w = \sqrt{ac - b^2}$
Eliminating ϕ we get the Beltrami equation

$$\left(\frac{a\bar{\psi}_x + b\bar{\psi}_y}{w} \right)_x + \left(\frac{b\bar{\psi}_x + c\bar{\psi}_y}{w} \right)_y = 0$$

 (doesn't really simplify the problem)

- Wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$



with $\xi = x + ct, \eta = x - ct \Rightarrow (u_\xi)_\eta = 0$

integrate twice, solve linear system to get

D'Alembert's solution :

$$u(x, t) = \frac{1}{2} \left(f(x+ct) + f(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

$\xleftarrow{\text{left-going}}$ $\xrightarrow{\text{right-going}}$

Ex: Vibrating string
boundary cond.



- Solve with Sep. Variables + Fourier series (sine series)

$$u(x, 0) = f(x) \text{ initial displacement}$$

$$u_t(x, 0) = g(x) \text{ initial speed.}$$

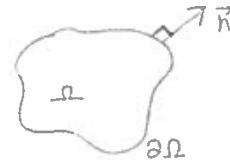
- When $g(x) = 0$ we can say that we abandoned the wave with initial profile $f(x)$.

- a standing wave = 2 synchronized/combined waves
(stationary wave) (in opposite directions)

① Laplace's Equation

Let $u(x) = u(x_1, x_2, \dots, x_n) \in C^2(\Omega)$

$$\text{with } \Delta \equiv \sum_{i=1}^n D_i^2, \quad D_i \equiv \frac{\partial}{\partial x_i}$$



Let Ω = open set where the divergence theor. can be applied. Take $u, v \in C^2(\bar{\Omega})$, meaning C^2 all the way to the boundary. Green's identity (int by parts) yields

$$\int_{\Omega} v \Delta u \, dx = - \int_{\Omega} \sum_i v_i u_{x_i} \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS$$

and

$$\int_{\Omega} v \Delta u \, dx = \int_{\Omega} u \Delta v \, dx + \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, dS.$$

In the special case $\boxed{v \equiv 1}$:

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS$$

This is useful for Poisson's eq $\Delta u = f$ with Neumann data on the boundary: $\frac{\partial u}{\partial n} = g(x), x \in \partial\Omega$.

This is a compatibility condition:

$$\int_{\Omega} f(x) \, dx = \int_{\partial\Omega} g(x) \, dS$$

Lipschitz domain = domains whose bdry is locally described by the graph of a Lipschitz function

$$|u(x) - u(y)| \leq L |x - y|$$

$$\frac{|u(x) - u(y)|}{|x - y|} \leq L, \text{bdd increments.}$$

u Lipschitz, then u is differentiable except at a set of measure zero.

Divergence theorem

$$\int_{\Omega} \nabla \cdot \vec{F} dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n} ds$$



$$\vec{F} = \nabla u$$

$$\int_{\Omega} \nabla \cdot \nabla u dx = \int_{\Omega} \Delta u dx = \int_{\partial\Omega} \frac{du}{dn} ds$$

$$\int_{\Omega} v \nabla \cdot \vec{F} dx = \int_{\Omega} \nabla \cdot (v \vec{F}) dx - \int_{\Omega} \vec{F} \cdot \nabla v dx$$

$$\int_{\Omega} v \nabla \cdot \vec{F} dx = \int_{\partial\Omega} v \vec{F} \cdot \vec{n} ds - \int_{\Omega} \vec{F} \cdot \nabla v dx$$

$$\vec{F} = \nabla u,$$

$$\int_{\Omega} v \Delta u dx = \int_{\partial\Omega} v \frac{du}{dn} ds - \int_{\Omega} \nabla u \cdot \nabla v dx$$

switch $u \leftrightarrow v$.

\Rightarrow Green's id.

Change role of u and v

51.

C

$$\int_{\Omega} u \Delta v = \int_{\partial\Omega} u \frac{dv}{dh} - \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

Subtract 2 above

$$\int_{\Omega} v \Delta u \, dx - \int_{\Omega} u \Delta v \, dx = \int_{\partial\Omega} \left(v \frac{du}{dh} - u \frac{dv}{dh} \right) ds$$

- In Fluids, heat problems etc. where we have Poisson's eq. this is a balance between the total source inside and a flux along the boundary.
- forcing term

Another case for Green's identity is when $u = v$:

(intermediate step
before \square)

$$\int_{\Omega} \sum u_{xi}^2 dx + \int_{\Omega} u \Delta u dx = \int_{\partial\Omega} u \frac{du}{dn} dS$$



Potential theory (Laplace's eq.) has 2 classical problems:

Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f(x) & \text{on } \partial\Omega \end{cases}$$

Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{du}{dn} = g(x) & \text{on } \partial\Omega \end{cases}$$

useful version for uniqueness.

Checking:

If $\Delta u_1 = w$ & $\Delta u_2 = w$ are two solutions
for the same boundary data, then $\begin{cases} \Delta(u_1 - u_2) = 0 \text{ in } \Omega \\ (u_1 - u_2) = 0 \text{ on } \partial\Omega \end{cases}$

$$\# \Rightarrow \int_{\Omega} \sum_i (u_1 - u_2)_x^2 dx = 0 \Rightarrow \text{uniqueness. (check)}$$

Note: Neumann case: u uniqueness up to a constant.

In some cases this constant does not matter

Ex: Fluids where the velocity potential $\phi(x, y)$,
 $(u(r)) = \nabla \phi$ satisfies $\Delta \phi = 0$, $\frac{d\phi}{dn} = 0$ on $\partial\Omega$ (no flow across $\partial\Omega$)
and

and is invariant under translation.

Laplace eq. has spherical symmetry. Write it in spherical coordinates and keep only the radial dependence.
Namely define

$$v = \psi(r) \quad r = |\mathbf{x} - \mathbf{z}| = \sqrt{\sum_i (z_i - x_i)^2}, \quad \left\{ \begin{array}{l} \mathbf{x}, \mathbf{z} \in \mathbb{R}^n \\ \text{centered at } \mathbf{x} = \mathbf{z} \end{array} \right.$$

which solves Laplace's eq. in dimension-n:

$$\Delta v = \psi''(r) + \frac{n-1}{r} \psi'(r) = 0.$$

This is the ODE

$$w'(r) = \frac{1-n}{r} w(r) \Rightarrow w(r) = C r^{(1-n)}$$

Where its solution $\omega(r) = \Psi'(r)$ is integrated to give .

$$\Psi'(r) = \begin{cases} C \ln r, & n=2 \\ C \frac{r^{2-n}}{2-n}, & n>2 \end{cases}$$

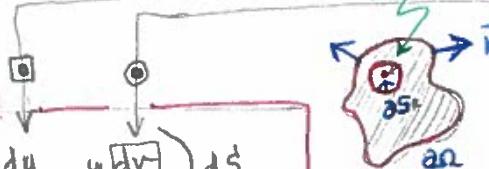
This function has a singularity at $x=3$. Let's check the nature of this singularity and correspondingly find a value for C .

- Take a ball centered at $x=3$ having a small radius $r=\varepsilon \ll 1$. Use Green's identity in the domain

$$\Omega_\varepsilon = \Omega - B(3, \varepsilon)$$

$$\int_{\Omega_\varepsilon} (r \Delta u + u \Delta v) dx = \int_{\partial \Omega_\varepsilon} \left(r \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dS,$$

will shrink hole



Where S_ε is the spherical surface of radius ε .

Let $\begin{cases} u \text{ is smooth} \\ \text{at least } 2 \end{cases}$ & $v = \Psi(r)$. On the spherical contour S_ε

$$r = \Psi(\varepsilon), \quad \frac{dv}{dr} = \Psi'(\varepsilon) = C \varepsilon^{(1-n)}$$

Also note that

$$\int_{S_\varepsilon} r \frac{\partial u}{\partial n} dS = \Psi(\varepsilon) \int_{S_\varepsilon} \frac{du}{dn} dS = \Psi(\varepsilon) \left(- \int_{B(3, \varepsilon)} \Delta u dx \right)$$

Div. Th

and from ① above

$$\text{② } \int_{S_\epsilon} u \frac{dv}{dn} dS = -G \epsilon^{(1-n)} \int_{S_\epsilon} u dS.$$

- Using the smooth properties of u

we have for $\boxed{\epsilon \ll 1}$:

$$\left\{ \begin{array}{l} \text{① } \int_{S_\epsilon} v \frac{du}{dn} dS \approx -\bar{\psi}(\epsilon) \Delta u(\bar{z}) \mathcal{O}(\epsilon^n) = \Delta u(\bar{z}) \mathcal{O}(\epsilon^2) \\ \quad \downarrow \quad \downarrow \\ \text{value at center} \quad \text{volume of ball} \\ \quad \downarrow \quad \downarrow \\ \text{② } \int_{S_\epsilon} u \frac{dv}{dn} dS \approx -G \epsilon^{(1-n)} \boxed{u(\bar{z}) w_n \epsilon^{(n-1)}} \quad \boxed{w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \text{ area of unit sphere}} \\ \quad \uparrow \quad \uparrow \\ \text{extracting the singular effect of } \bar{\psi} \quad \text{value at center} \end{array} \right. \quad \text{will NOT contribute.}$$

- Take the limit as $\epsilon \rightarrow 0$:

$$\textcircled{1} \Rightarrow \int_{\Omega} v \Delta u dx = \underbrace{\int_{\partial\Omega} \left(v \frac{du}{dn} = u \frac{dv}{dn} \right) dS}_{\Delta u = 0 \text{ in } \Omega_\epsilon} + \boxed{\bar{\psi}(\bar{z})},$$

where we have used: $\boxed{G = 1/w_n}$. Take a test function

$u = \phi \in C_0^\infty(\Omega)$ to conclude that

$$\boxed{\int_{\Omega} v \Delta \phi dx = \phi(\bar{z})} \Leftrightarrow \boxed{\Delta v[\phi] = \delta_{\bar{z}}}$$

$$v = \bar{\psi}(\bar{r}), \quad G = 1/w_n,$$

\hookrightarrow the fundamental solution of Laplace's eqn
(is a generalized funct.)

- $v =$ is the fundamental solution of the Laplacian.
the (free space) Green's function.
 \hookrightarrow (usually associated w/ homogeneous
bdry cond.)

- Different notations

$$v = K(x, \xi) = \tilde{\Psi}(|x - \xi|) = \tilde{\Psi}(r)$$

- This weak solution to the Laplacian, in the sense of distributions, is defined as

$$\int_{\Omega} K(x, \xi) \Delta \phi(x) dx = \phi(\xi)$$

for all test functions
 $\phi(x) \in C_0^\infty(\Omega)$,
 will comment more
 below.

Note that for $n=2$ the singularity is logarithmic,
 not as "strong" as for $n \geq 3$.

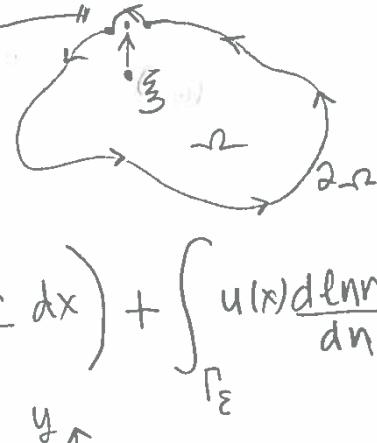
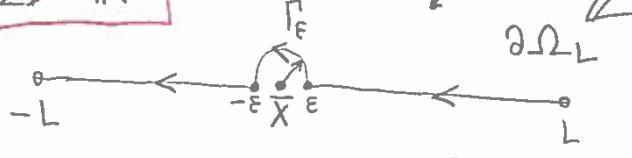
- Green's identity with $u \in C^2(\bar{\Omega})$, $\Delta u = 0$ and
 $v = K(x, \xi)$, $\Delta v = \delta_\xi$ yields (Green's third identity)

$$u(\xi) = - \int_{\partial\Omega} \left(K(x, \xi) \frac{du}{dn}(x) - u(x) \frac{dK}{dn}(x, \xi) \right) dS_x, \quad x \in \partial\Omega, \quad \xi \in \Omega$$

Note: if we know both Neumann data and Dirichlet data
 single layer potential source density dipole density (moment density)
 double layer potential

$$2D \equiv \mathbb{R}^2$$

Schematics in a simple configuration



$$\int_{\partial\Omega_L} u(x) \frac{d\ln r}{dn} dx = \left(\int_{-L}^L + \int_{-\varepsilon}^{-L} u(x) \frac{d\ln r}{dn} dx \right) + \int_{\Gamma_\varepsilon} u(x) \frac{d\ln r}{dn} ds$$

From complex variables we know that

$$\log z = \ln r + i \arg z$$

$$\theta = \arg z = \arctan \frac{y}{x}$$

$$z = r e^{i\theta}$$

We also know that

$$\log z = u(x, y) + i v(x, y)$$

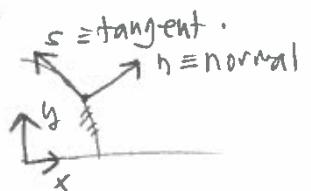
where v is the harmonic conjugate of u

with

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

or also

$$\begin{aligned} u_n &= v_s \\ u_s &= -v_n \end{aligned}$$



Therefore $\arctan(y/x)$ is the harmonic conf. of $\ln r$, $r = \sqrt{x^2 + y^2}$. Using this fact

$$\begin{aligned} \int_{\Gamma_\varepsilon} u(x) \frac{d\ln r}{dn} ds &= \int_{\Gamma_\varepsilon} u(x) \frac{d\arctan y/x}{ds} ds = \\ &= u(\bar{x}) \int_0^\pi \frac{d\arctan(y/x)}{ds} ds + \Theta(\varepsilon) = \pi u(\bar{x}) + \Theta(\varepsilon) \end{aligned}$$

$\varepsilon \downarrow 0$

$$\Theta u(\bar{x}) = \int_{\partial\Omega} \left(u(x) \frac{d\mathbb{B}(x, \bar{x})}{dn} - \frac{du}{dn}(x) \mathbb{B}(x, \bar{x}) \right) ds$$

Functional relation Dirichlet-Neumann

Along the lines of 1/2 residue: $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ w/ complex variables

$x \in \partial\Omega$
 $\theta = \pi, \bar{x} \in \partial\Omega$ smooth
 $\theta = \text{internal angle}$
 $\partial\Omega$ -corner

Weak derivatives & test functions.

Let $\phi(x) \in \mathbb{D} = C_0^\infty(\Omega)$ be called test function.

↳ has derivatives of all orders.

↳ compact support: zero outside a closed bounded set.

Take

$$\boxed{\int_{\Omega} \frac{df}{dx} \phi dx = - \int_{\Omega} f \frac{d\phi}{dx} dx}, \text{ since } \text{supp}(\phi) \subset \Omega$$

When $f \in C^1(\Omega)$ this is a trivial identity.

But when the derivative of f does NOT make sense we use the expression above to say that ...

... v is the weak derivative of f (derivative in a generalized sense)

If



$$\boxed{\int_{\Omega} v \phi dx = - \int_{\Omega} f \frac{d\phi}{dx} dx, \quad \forall \phi \in C_0^\infty.}$$

for all test functions

Weak derivative is done in a nonlocal fashion.
(Derivative in the sense of distributions - by L. Schwartz)

We can define a functional with f as

$$f[\phi](x) = \int_{\mathbb{R}} f(x) \phi(x) dx , \text{ for all } \phi \in \mathcal{D} = C_0^\infty(\mathbb{R})$$

We can also define a functional

$$\frac{d}{dx} f[\phi](x) = - \int_{\mathbb{R}} f(x) \left(\frac{d\phi}{dx} \right) dx , \text{ for all } \phi \in \mathcal{D}$$

which is the derivative of f in the sense of distributions
bearing in mind ϕ .

- Ex: $\delta_\xi[\phi] = \phi(\xi)$ Dirac's delta function $\int_{\mathbb{R}} \delta_\xi dx = 1$
 ↑ generalized function in the sense of distributions
- Ex: weak solution of PDE obtained through.

$$Lu[\phi] = u[\tilde{L}\phi] = \int_{\mathbb{R}} u(\tilde{L}\phi) dx$$

↑
adjoint of L .
Have all weak derivatives here.

- Ex: $L = \Delta = \tilde{L}$ (see page 55)
 We have that the Green's function $v = R(x, \xi)$ satisfies

$$\boxed{\Delta v[\phi] = \delta_\xi}$$

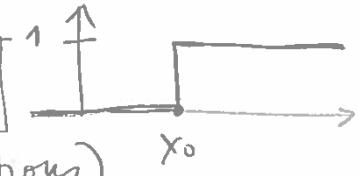
$$R(x, \xi) = R_\xi(x)$$

$$\Delta R_\xi[\phi](x) = \delta_\xi[\phi](x)$$

Example of weak derivative:

① Heaviside function

$$f(x) = H(x - x_0)$$



in the sense of weak derivatives (distributions)

$$\frac{d}{dx} f[\phi](x) = - \int_{-\infty}^{\infty} H(x - x_0) \frac{d\phi}{dx} dx, \quad \phi \in C_0^\infty(\mathbb{R})$$

$$= - \int_{x_0}^{\infty} \frac{d\phi}{dx} dx = -(0 - \phi(x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) \phi(x) dx$$

true for all
test functions

We conclude that

$$\frac{d}{dx} f[\phi](x) = \delta_{x_0}[\phi](x)$$

② $g(x) = a H(x - x_0)$; informally we can just write

$$\frac{dg}{dx} = a \delta(x - x_0)$$

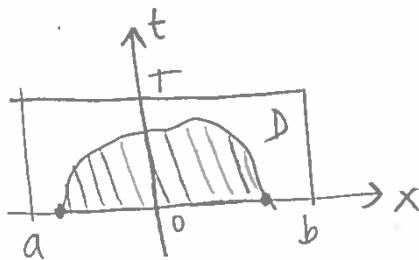
↑ intensity of the Dirac delta function.

↑ rate of change of g ↑ "pushes" g from "level zero" to "level a ".
(the impulse)

C) Back to Conservation Law:

$$\textcircled{X}_1 \quad \text{take } \begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x) \end{cases}$$

If solution is classical $C^1(\mathbb{R})$ and $\phi(x, t)$ is a test function with support in $D = [a, b] \times [0, T]$



Then

$$\begin{aligned} \int_0^T \int_a^b (\phi u_t + \phi f(u)_x) dx dt &= 0 \\ \int_a^b \int_0^T [(\phi u)_t - u \phi_t] dt dx + \int_0^T \int_a^b [(\phi f(u))_x - \phi_x f(u)] dx dt &= \\ - \int_a^b \int_0^T u \phi_t dt dx + \int_a^b (\phi u)_0^T - \int_a^b \int_0^T \phi_x f(u) dt dx + \int_0^T (\phi f(u))_a^b &= \end{aligned}$$

$$\boxed{\textcircled{X}_2 \quad \int_a^b \int_0^T (u \phi_t + f(u) \phi_x) dt dx + \int_a^b \phi(x, 0) u_0(x) dx = 0}$$

Then \textcircled{X}_2 is the WEAK FORM of \textcircled{X}_1 if it is satisfied for all test functions ϕ .

Note: One can recover the Rankine-Hugoniot cond. from this weak form.

- Is Green's func. unique?

let

$$\Delta u = 0 \quad u(x) \in C^2(\bar{\Omega}), x \in \mathbb{R}^n$$

then $G(x, \xi) = \Phi(x, \xi) + w(x)$ is also a fundamental solution

- Consider a simple domain : $\Omega = B(3, p)$ ball centred at $x=3$
radius = p .

$$G(x, \xi) = \Psi(|x - \xi|) - \Psi(p), \quad \Psi(p) = \text{constant}$$

$$G \equiv 0 \quad \text{on } \partial\Omega$$

works for a fixed ξ

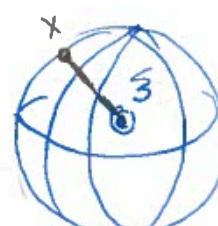
$$\frac{dG}{dn} = \Psi'(p) = \frac{1}{\omega_n} p^{1-n}$$

Substitute in Green's identity

$$u(3) = \int_{\Omega} G(x, \xi) \Delta u \, dx - \int_{\partial\Omega} \left(G(x, \xi) \frac{du}{dn} - u \frac{dG}{dn}(x, \xi) \right) dS$$

for harmonic functions we obtain

Green's Law of the arithmetic mean:



$$u(3) = \frac{1}{\omega_n p^{n-1}} \int_{\partial\Omega} u(x) dS,$$

$\omega_n p^{n-1} \equiv \text{area of ball } B(3, p)$

- There are different versions - below is a strong version. 62

Max. Principle: $\Omega = \text{open, bounded, connected in } \mathbb{R}^n$

let $u \in C^2(\Omega)$ with $\Delta u = 0$ in Ω .

then

either u is a constant

or the maximum is at the boundary.

- Note 1: this is clearly a consequence of Gauss' law (above).

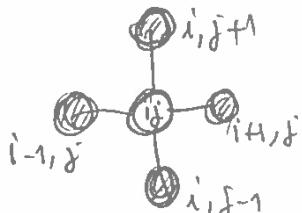
- Note 2: We immediately have for a harmonic function a minimum principle.

take $v = -u$, $\Delta v = 0$

$\max v \Rightarrow \min u$.

- Note 3: Max principle is another way to conclude uniqueness for the Dirichlet problem.

- Note 4: Take the five-point stencil for the discrete Laplacian



$$\begin{aligned} & u(x-\Delta x, y) + u(x+\Delta x, y) + u(x, y+\Delta y) + \\ & + u(x, y-\Delta y) - 4u(x, y) \approx h^2 \Delta u(x, y) \end{aligned}$$

What do we see? $h = \Delta x = \Delta y$

① Green's Function

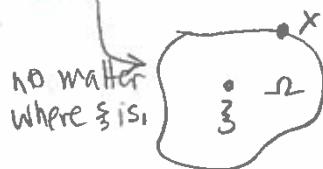
We saw that ($\xi \in \Omega$)

$$u(\xi) = \int_{\Omega} G(x, \xi) \frac{\partial u}{\partial n} dx - \int_{\partial\Omega} \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) dS_x,$$

where u depends on the Cauchy data $(u, \frac{\partial u}{\partial n})$; that can not be imposed arbitrarily) and on the fundamental solution G . Let $x \in \partial\Omega$, $\xi \in \Omega$ and $G(x, \xi) = 0$ be the Green's function in this domain Ω .

Then

$$u(\xi) = \int_{\partial\Omega} f(x) \frac{dG(x, \xi)}{dn} dS_x$$



→ How to construct such a $G(x, \xi)$?

Define

$$G(x, \xi) = K(x, \xi) + v(x, \xi)$$

#

$K(x, \xi) = \frac{1}{4\pi} \ln |x - \xi|$ as before with $v(x, \xi)$ given by

$$\Delta v(x, \xi) = 0, \quad \xi \in \Omega \text{ with } v \in C^2(\bar{\Omega}),$$

as a parameter.

- this notation allows for v to be another fundamental solution but with the singularity outside Ω as we will see.
- We want

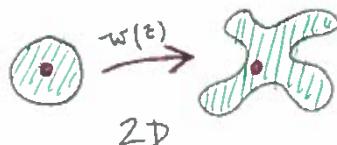
$$\boxed{G(x, \bar{z}) = 0, \quad x \in \partial\Omega, \quad \bar{z} \in \Omega} \quad \otimes$$

So we need to ask for (in order to have \otimes)
using $\#$

$$\boxed{\begin{aligned} \Delta_x v &= 0, \quad x \in \Omega, \quad \bar{z} \in \Omega. \\ v(x, \bar{z}) &= -K(x, \bar{z}), \quad x \in \partial\Omega. \end{aligned}}$$

-
- this problem can be solved explicitly in some cases as for example the sphere or the half-plane.
We use a reflexion principle.
 - Note: In 2D we can use conformal mapping to map the disk or half-plane onto other simply connected regions.

this is guaranteed by the Riemann mapping theorem.



① the case of the Sphere :

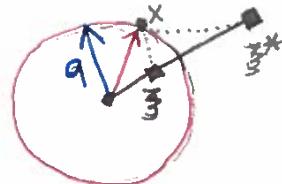
let the ball be given by

$\Omega = B(0, a) = \{x \mid |x| < a\}$. Take a point inside the ball

$\vec{z} \in \Omega$ and let $\vec{z}^* = \frac{a^2}{|\vec{z}|^2} \vec{z}$ be its REFLEXION with respect

to the spherical shell :
(boundary)

Due to the fundamental solution we
ask for:



$$\frac{r}{r^*} = \frac{|x - \vec{z}|}{|x - \vec{z}^*|} = \text{constant} \quad \square_1, \quad x \in \partial\Omega.$$

This leads to the result that the constant is

$$\frac{r^*}{r} = \frac{a}{|\vec{z}|} = \text{constant} \quad \square_2$$

and the reflection (image)
point is \vec{z}^* above, which

depends of $x \in \partial\Omega$.

For $n > 2$ write

$$B(x, \vec{z}) = \frac{1}{(2-n)\omega_n} r^{2-n}$$

$$B(x, \vec{z}^*) = \frac{1}{(2-n)\omega_n} r^{*2-n}$$

$$r^* = \frac{a}{|\vec{z}|} r$$

and when

$x \in \partial\Omega$,

using $\square_1 + \square_2$

$$B(x, \vec{z}^*) = \left(\frac{a}{|\vec{z}|}\right)^{2-n} B(x, \vec{z})$$

$$\text{if } v(x, \vec{z}^*) = -\left(\frac{1}{|\vec{z}|}\right)^{2-n} K(x, \vec{z}^*)$$

then $\Delta v = 0$ in Ω

and $v = -K(x, \vec{z}) @ \partial\Omega$

④ Finding the reflexion point (image).

$$\frac{q}{4\pi|\xi^*-x|} = \frac{1}{4\pi|\xi-x|}$$

when $|x|=a$
 $q=\text{some constant}$

$$q|\xi-x| = |\xi^*-x|$$

$$q^2|\xi-x|^2 = |\xi^*-x|^2$$

$$q^2(|\xi|^2 - 2\xi \cdot x + a^2) = |\xi^*|^2 - 2\xi^* \cdot x + a^2$$

$$\begin{aligned} q^2|\xi|^2 + q^2a^2 - |\xi|^2 - a^2 &= 2q^2\xi \cdot x - 2\xi^* \cdot x \\ &= 2x \cdot (q^2\xi - \xi^*) \end{aligned}$$

Must be valid for all x .

LHS does not depend on x .

Therefore

$$q^2\xi - \xi^* = 0 \quad \rightarrow \quad \boxed{\xi^* = q^2\xi}$$

$$|\xi^*|^2 - q^2(|\xi|^2 + a^2) + a^2 = 0$$

$$q^4|\xi|^2 - q^2(|\xi|^2 + a^2) + a^2 = 0$$

$$q^2 = \frac{(|\xi|^2 + a^2) \pm \sqrt{(|\xi|^2 + a^2)^2 - 4|\xi|^2 a^2}}{2|\xi|^2}$$

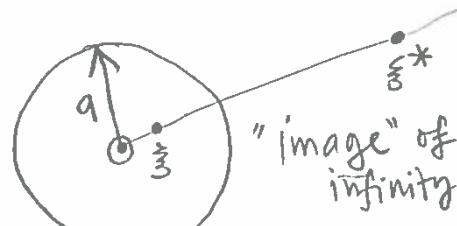
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$$\textcircled{C} \quad q^2 = \frac{(|\beta|^2 + a^2) \pm \sqrt{(|\beta|^2 - a^2)^2}}{2|\beta|^2}$$

$$\left\{ \begin{array}{l} q^2 = \frac{2|\beta|^2}{2|\beta|^2} = 1 \\ q^2 = \frac{2a^2}{2|\beta|^2} = \frac{a^2}{|\beta|^2} \end{array} \right.$$

$$\boxed{\beta^* = a^2 \frac{\beta}{|\beta|^2}}$$

In other words



"image" of zero is infinity.

$$z^* = \frac{a^2}{|z|^2}$$

$+r(x,z)$, $\Delta v=0$ (page 63)

$$G(x, z) = K(x, z) = \left(\frac{|z|}{a} \right)^{2-n} K(x, z^*)$$

Singularity is outside.

where $\begin{cases} \Delta G = \delta_z \text{ in } \Omega \\ G(x, z) = 0, x \text{ on } \partial\Omega \end{cases}$

As we saw before, for u harmonic in $\Omega \equiv B(0, a)$

$$u(z) = \int_{|x|=a} u \frac{dG}{dn} dS_x$$

and this leads to Poisson's integral formula

$$u(z) = \int_{|x|=a} H(x, z) f(x) dS_x$$

$$, |z| \leq a, \\ u(x) = f(x), x \in \partial\Omega$$

$$H(x, z) = \frac{1}{2\pi n} \frac{a^2 - |z|^2}{|x-z|}$$

Poisson kernel.

In 2D : this formula can be obtained from Complex Variable, using Cauchy

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$$

boundary integral representation for 1/2 harmonic functions

① Verification

Given

$$\Delta u = 0$$

$$u(x) = f(x), x \in \partial\Omega$$

Is the function represented by

$$u(\xi) = \int_{|x|=a} H(x, \xi) f(x) dS_x, \quad |\xi| < a,$$

harmonic in $\Omega : |\xi| < a$, C^∞ in $\Omega : |\xi| < a$
and continuous in $|\xi| \leq a$ (all the way to the bdry) ?

Most points are easy to check through the Poisson kernel

smooth func, bdd domain

(a) $H(x, \xi) \in C^\infty$ in ξ ; $|x| \leq a, |\xi| < a, \boxed{x \neq \xi}$

(b) $\Delta_\xi H(x, \xi) = 0, \quad \rightarrow |\xi| < a, |x| = a, \quad \text{at } \xi$

(c) $\int_{|x|=a} H(x, \xi) dS_x = 1, \quad |\xi| < a, \quad \text{at } \xi$ } "weight function"
total mass = 1

(d) $H(x, \xi) > 0, \text{ for } |x| = a, |\xi| < a.$

(e) When $|\xi| = a$, then $\lim_{\substack{\xi \rightarrow \xi \\ |\xi| < a}} H(x, \xi) = 0$ uniformly in x

for x such that $|x - \xi| > \delta > 0$

$$H(x, \xi) = \frac{1}{a \sinh \frac{a^2 - |\xi|^2}{2|x - \xi|}}, \quad |\xi| < a.$$

- Checking

(a), (b), (d) and (e) are very easy from the Poisson kernel's expression.

(c) : has a nice trick. Poisson formula is valid for the harmonic function $u(x) \equiv 1$.
Then $f(x) = 1$ and $u(\xi) = 1$.

(Expr. from Green's id.)

- We have that u is C^∞ and u is harmonic in $|\xi| < a$.

- Check that u is continuous up to the boundary and satisfies the Dirichlet boundary condition: this is a little trickier.

Set $|z|=a$, $|\xi| < a$. We want to show that as $\xi \rightarrow z$ this difference can be made smaller and smaller, despite

$$\boxed{u(\xi) - f(\xi)} = \int_{|x|=a} H(x, \xi) f(x) dS_x - \int_{|x|=a} f(\xi) H(x, \xi) dS_x =$$

rewritten
 \downarrow
 $\equiv I_1 + I_2$,

common trick of writing $f(\xi)$ in a form but useful way.

$$I_1 = \int_{|x|=a} H(x, \xi) (f(x) - f(\xi)) dS_x$$

$$|x-\xi| < \delta$$

$$I_2 = \int_{|x|=a} H(x, \xi) (f(x) - f(\xi)) dS_x$$

$$|x-\xi| > \delta$$

where

lets try to control these integrals, near and away from the singularity at \bar{z} . I_1 I_2

Pick a tolerance $\varepsilon > 0$ and its related $\delta = \delta(\varepsilon)$ as in

near $|I_1| \leq \int H(x, \bar{z}) |f(x) - f(\bar{z})| dS_x \leq \varepsilon \cdot 1$

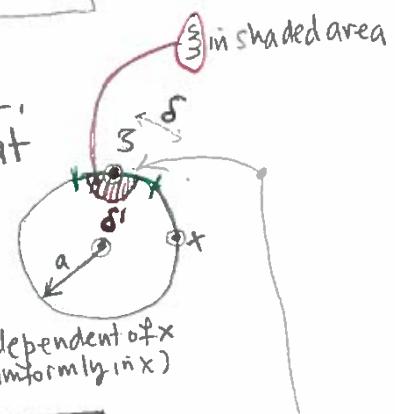
• $|\int f(x) dS_x| \leq \int |f(x)| dS_x$

• by Continuity of f : $|x - \bar{z}| < \delta(\varepsilon), |x| = a \Rightarrow |f(x) - f(\bar{z})| < \varepsilon$

• by (c) and (d): $\int |H| dS_x = \int H dS_x = 1$

Now I_2 : Let $M = \max |f(x)|, x \in \partial\Omega$. Given tolerance ε' , by (e) can find $\delta'(\varepsilon')$ such that

$$H(x, \bar{z}) \leq \frac{\varepsilon}{M \ln a^{n-1}} = \varepsilon'$$



So When $|\bar{z} - \bar{z}| < \delta'(\varepsilon')$ and $|x - \bar{z}| > \delta(\varepsilon)$.

shaded area $|I_2| \leq \int_{|x|=a} |f(x) - f(\bar{z})| H(x, \bar{z}) dS_x \leq 2M \left(\frac{\varepsilon}{M \ln a^{n-1}} \right) (\ln a^{n-1}) = 2\varepsilon.$

We conclude from $|I_1 + I_2| \leq |I_1| + |I_2| \leq 3\varepsilon$ that

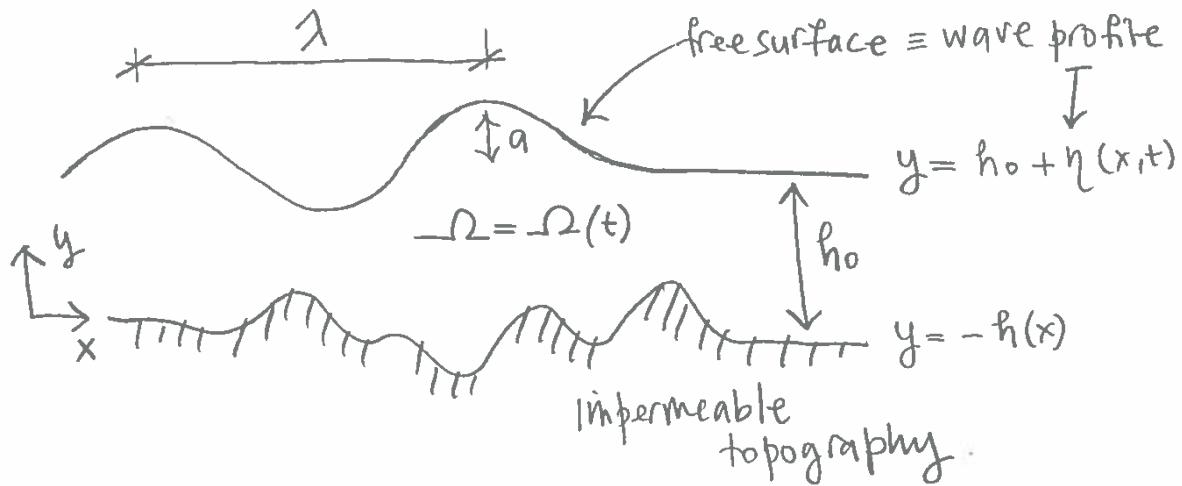
from $|u(\bar{z}) - f(\bar{z})| < 3\varepsilon$, when $|\bar{z} - \bar{z}| < \delta'$ $|\bar{z}| \rightarrow a$

given this tolerance/error

as small as we want, we can find a neighborhood δ' (shaded area above) of \bar{z} such that when \bar{z} is in the neighborhood

$$u(\bar{z}) \approx f(\bar{z}) = \text{Dirichlet data}$$

Water Waves in an incompressible,
irrotational regime



Physical
Modeling point of view:

- viscous effects are ignored: no bottom friction.
no vorticity generation.
- fixed impermeable bottom:
 - no sediment transport
 - bottom porosity ignored - no seepage effect
- problem of interest from BOTH physical (applications)
and mathematical points of view.

Mathematical modeling

2D - nonlinear potential theory.

- a free boundary problem.

use a velocity potential $\phi(x, y, t)$ where

$$(u, v) = \text{velocity} = \nabla \phi$$

- irrotational: $v_x - u_y = \phi_{yx} - \phi_{xy} = 0$

→ - incompressible: $u_x + v_y = 0 \Rightarrow \boxed{\Delta \phi = 0}$

- impermeable bottom: no flow

$$(u, v) \cdot \vec{n} = 0$$

$$\nabla \phi \cdot \vec{n} = \boxed{\frac{d\phi}{dn} = 0} \quad \text{Neumann cond.}$$

Dirichlet data

(Free boundary)

Free surface condition

evolution
thru the
boundary
(NONLINEAR)

$$\begin{cases} \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0, & \phi(x, h_0 + \eta_0(x), 0) = \phi_0(x) \text{ (Bernoulli Law)} \\ \eta_t + \phi_x \eta_x - \phi_y = 0, & \eta(x, 0) = \eta_0(x) \text{ (Kinematic Condition)} \end{cases}$$

a nonlocal component

$$\boxed{\Delta \phi = 0, \text{ in } \Omega = \Omega(t)}$$

$$y = h_0 + \eta(x, t) \quad \text{unknown}$$



$$y = -h(x)$$

$$\frac{d\phi}{dn} = 0 \text{ (Neumann cond.)}$$

— Many interesting problems in PDEs & Applications arise from the system above. Many important PDEs can be obtained from this system: the wave equation, the KdV eq., Boussinesq system, nonlinear Schrödinger eq. among others!
 HOW? Start by taking **DIMENSIONLESS VARIABLES** (\tilde{v})

$$x = \lambda \tilde{x}, \quad y = h_0 \tilde{y}, \quad t = \frac{\lambda}{c_0} \tilde{t}$$

$$\eta = a \tilde{\eta}, \quad \phi = \frac{g \lambda a}{c_0} \tilde{\phi}, \quad h = h_0 \tilde{h}\left(\frac{\lambda}{\ell} \tilde{x}\right)$$

Where

λ = reference / typical wavelength

h_0 = depth

c_0 = propagation speed

a = amplitude

ℓ = bottom variations' length scale

$$\frac{\lambda}{c_0} = \frac{[L]}{[L]/[T]} = [T] = \text{reference time}$$

reference potential

$$\hookrightarrow \frac{g \lambda a}{c_0} = \frac{[L]/[T]^2 \cdot [L] \cdot [L]}{[L][T]/[T]} = \frac{[L]^2}{[T]} \quad \leftarrow$$

checking: $\nabla \phi = \text{speed} \Rightarrow [\phi] = [\text{speed}] \cdot [L]$

$$[\nabla \phi] = \frac{[\phi]}{[L]}$$

$$\left[\frac{g \lambda a}{c_0} \right] = \frac{[L]}{[T]} \cdot [L] = [\phi] \quad \left(\frac{g \lambda a}{c_0} \right)$$

- Substitute in the potential theory equations and drop the tilde $\tilde{\cdot}$:

$\beta \phi_{xx} + \phi_{yy} = 0, \quad -h\left(\frac{x}{r}\right) \leq y \leq 1 + \alpha h$	
$\eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_y = 0$	free surface conditions
$\eta + \phi + \frac{1}{2} \alpha \phi_x^2 + \frac{1}{2} \frac{\alpha}{\beta} \phi_y^2 = 0$	
$\phi_y + \frac{\beta}{r} h'\left(\frac{x}{r}\right) \phi_x = \boxed{\frac{d\phi}{dn} = 0} \quad \text{bottom boundary condition}$	$y = -h\left(\frac{x}{r}\right)$
$\begin{cases} \phi(x_0 + dh_0, 0) = \phi_0(x) \\ \eta(x_0, 0) = \eta_0(x) \end{cases} \quad \text{initial conditions}$	

$\alpha = \frac{a}{h_0} = \text{nonlinearity parameter}$

$\beta = \frac{h_0^2}{\lambda^2} = \text{dispersion parameter (long waves/shallow water when small)}$

$\gamma = \frac{l}{\lambda} = \text{wave/bottom interaction parameter}$

Note: when $\alpha = 0$ \Rightarrow no nonlinear terms in PDE
 \Rightarrow not a free boundary problem anymore

"another source of nonlinearity"

REDUCED MODELING

- systematic simplification of PDES.
- ↓
- of what happens in the limit $\alpha, \beta \rightarrow 0$?
- " " " " the regime $\alpha = \Theta(\varepsilon), \beta = \Theta(\varepsilon), \varepsilon \ll 1$?
- First exercise in this direction. For simplicity take a FLAT BOTTOM.
- shallow waters/long waves $\Rightarrow \beta \ll 1$

Weakly nonlinear waves $0 < \alpha \ll 1$ \Rightarrow

in Shallow water regime the vertical structure of solution
changes very little. ◻

lets make an expansion about the bottom in the form
 $(y=0)$ -flat

$$\phi(x, y, t) = \sum_{n=0}^{\infty} y^n f_n(x, t)$$

lets first solve Laplace's equation and the (bottom)
Neumann condition in this power series form!

Laplace:

recurrence relation

$$\rho \phi_{xx} + \phi_{yy} = \sum_{m=0}^{\infty} y^m \left(\rho \partial_x^2 f_m + (m+2)(m+1) f_{m+2} \right) = 0$$

$$\Rightarrow \phi(x, y, t) = \left[\sum_{n=0}^{\infty} (-\beta)^n y^{2n} \frac{\partial_x^{2n} f_0}{(2n)!} \right] + \left[\sum_{n=0}^{\infty} (-\beta)^n y^{2n+1} \frac{\partial_x^{2n} f_1}{(2n+1)!} \right]$$

even terms odd terms

Neumann Cond.

$$\phi_y(x_1, 0, t) \equiv 0 \Rightarrow f_1(x_1, t) \equiv 0$$

\otimes_1

$$\phi(x_1, y_1, t) = \sum_{n=0}^{\infty} (-\beta)^n \frac{y_1^{2n}}{(2n)!} \partial_x^{2n} f(x_1, t)$$

Satisfies Laplace + Neumann

need no index.

- Now we need to deal with the free surface conditions which are to be solved at

$$y = 1 + \alpha \eta_0$$

from

\otimes_1

$$\left\{ \begin{array}{l} \phi_x = f_x - \frac{\beta}{2!} (1 + \alpha \eta_0)^2 f_{xxx} + \frac{\beta^2}{4!} (1 + \alpha \eta_0)^4 f_{xxxxx} - \dots \\ \phi_y = -\beta (1 + \alpha \eta_0) f_{xx} + \frac{\beta^2}{6} (1 + \alpha \eta_0)^2 f_{xxxxx} - \dots \\ \phi_t = f_t - \frac{\beta}{2} (1 + \alpha \eta_0)^2 f_{xxt} + \frac{\beta^2}{4!} (1 + \alpha \eta_0)^4 f_{xxxxxt} - \dots \end{array} \right.$$

- Substituting in the Bernoulli law and kinematic condition:

$$\left\{ \begin{array}{l} \eta_t + \alpha f_x \eta_x + (1 + \alpha \eta_0) f_{xx} - \frac{\beta}{6} f_{xxxxx} = \Theta(\beta^2, \alpha \beta, \alpha^2) \\ \eta + f_t - \frac{\beta}{2} f_{xxt} + \frac{\alpha}{2} f_x^2 = \Theta(\beta^2, \alpha \beta, \alpha^2) \end{array} \right.$$

terms we will discard

therefore called

weakly nonlinear
weakly dispersive

REGIMES
leading order in α & β .

- As our first approximation we have
(dropping quadratic terms in α and β)

$$\begin{cases} \eta_t + [(1+\alpha\eta) f_x]_x - \frac{\beta}{6} f_{xxxx} = 0 \\ \eta + f_t + \frac{\alpha}{2} f_x^2 - \frac{\beta}{2} f_{xx} t = 0 \end{cases}$$

leading order terms
in α and β

What is $f(x,t)$?
Physical meaning?
We will replace it below.

let

$$\tilde{u}(x,t) = f_x(x,t).$$

↓ slip velocity at the bottom

We used in Φ_x : $\Phi_x(x, I, t) = f_x(x, t) - \frac{\beta}{2} I^2 f_{xx}(x, t) + \Theta(\beta^2),$

where $I = 1 + \alpha \eta(x, t)$ is the free surface position.

Note that if we DEPTH AVERAGE (see comment □ page 74)

$$\frac{1}{I} \int_0^I \Phi_x(x, y, t) dy = \frac{1}{I} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!} \left[\int_0^I y^{2n} dy \right] \partial_x^{2n+1} f =$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!} \frac{I^{2n}}{2n+1} \partial_x^{2n+1} f$$

depth averaged
horizontal speed

$$\tilde{U}(x, t) = \tilde{u}(x, t) - \frac{\beta}{6} (1 + \alpha \eta)^2 f_{xx} + \Theta(\beta^2) =$$

$$= \tilde{u}(x, t) - \frac{\beta}{6} \tilde{u}_{xx}(x, t) + \Theta(\beta^2, \alpha \beta)$$

we want to invert roles in order to use / sub.
in PDEs.

We have that $\tilde{u} = u + \Theta(\beta)$ and

$$\tilde{u}(x,t) = u(x,t) + \frac{\beta}{6} u_{xx}(x,t) + \Theta(\beta^2, \alpha\beta)$$

- Substituting in the PDES and dropping terms of order $\Theta(\beta^2, \alpha\beta, \alpha^2)$:

$$\left\{ \begin{array}{l} \eta_t + \left[(1+\alpha\eta_b)(u + \frac{\beta}{6}u_{xx}) \right]_x - \frac{\beta}{6}u_{xxx} = 0 \\ \left[u + \frac{\beta}{6}u_{xx} \right]_t + \eta_{bx} + \alpha \left\{ \left[u + \frac{\beta}{6}u_{xx} \right] \left[u_x + \frac{\beta}{6}u_{xxx} \right] \right\} - \frac{\beta}{2}u_{xx}t = 0 \end{array} \right.$$

Note that we have differentiated this equation with respect to x in order to have speeds and not reduced potentials $f(x,t)$.

- Dropping higher order terms:

$$\left\{ \begin{array}{l} \eta_t + [(1+\alpha\eta_b)u]_x = 0 \\ u_t + \eta_{bx} + \alpha uu_x - \frac{\beta}{3}\eta_{bx}t = 0 \end{array} \right.$$

- From the first equation
We can use that $\eta_t = -u_x + \Theta(\alpha)$.
which allows to exchange x - and t -derivatives:

$$\boxed{\begin{array}{l} \eta_t + [(1+\alpha\eta_b)u]_x = 0 \\ u_t + \eta_{bx} + \alpha uu_x + \frac{\beta}{3}\eta_{bx}t = 0 \end{array}}$$

weakly nonlinear

weakly dispersive

Boussinesq system.
dimensionless form
Similar to original form

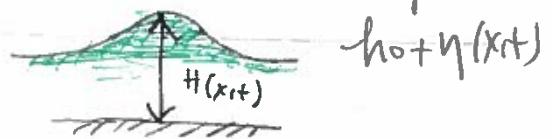
- In DIMENSIONAL variables the original system was given as

original dimensional
Boussinesq
equation (1872)

$$\begin{cases} H_t + (uH)_x = 0 \\ u_t + gH_x + uu_x + \frac{1}{3} h_0 H_{xtt} = 0 \end{cases}$$

u = horizontal speed
(average)

$H(x, t)$ = total water depth



Remark 1 : $\beta = 0$ regime + non dispersive regime

shallow water system
(page 42)

$$\begin{cases} H_t + (uH)_x = 0 \\ u_t + \left(gH + \frac{u^2}{2}\right)_x = 0 \end{cases}$$

$$\begin{cases} \eta_t + ((1+\alpha\eta)u)_x = 0 \\ u_t + \left(\eta + \alpha\frac{u^2}{2}\right)_x = 0 \end{cases}$$

or

dimensionless

$\alpha = 0 + \beta = 0$ linear hyperbolic

$$\tilde{u} = u + \frac{\beta}{6} u_{xx} + O(\beta^2, \alpha\beta)$$

$$\begin{cases} \eta_t + \tilde{u}_x = 0 \\ \tilde{u}_t + \eta_x = 0 \end{cases}$$

$\boxed{\eta_{tt} - \eta_{xx} = 0}$

Wave equation.

- How to extract a unidirectional weakly nonlinear, weakly dispersive wave equation (namely the KdV) from the Boussinesq system?
- Go back to version in page 76:

$$\Phi_1 \quad \begin{cases} \eta_t + [(1+\alpha\eta)\tilde{u}]_x - \frac{\beta}{6}\tilde{u}_{xxx} = \Theta(\alpha^2, \alpha\beta, \beta^2) \\ \tilde{u}_t + \alpha\tilde{u}\tilde{u}_x + \eta_x - \frac{1}{2}\beta\tilde{u}_{xxt} = \Theta(\alpha^2, \alpha\beta, \beta^2) \end{cases}$$

- In the linear case we know that we can choose "uni-directional"-data for

$$\begin{cases} \eta_t + \tilde{u}_x = 0 \\ \tilde{u}_t + \eta_x = 0 \end{cases}$$

Let $\tilde{u}_0 = \eta_0$ $\Rightarrow \begin{cases} \eta_{xt} + \eta_t = 0 \Rightarrow \eta(x,t) = \eta_0(x-t) \text{ (CHECK!)} \\ \eta_t = u \leftarrow \end{cases}$

- this provides us with an ansatz:

$$\boxed{\tilde{u} = \eta + \alpha A + \beta B + \Theta(\alpha^2, \beta^2)}$$

\otimes_1 \otimes_2 \otimes_3

Φ_2

\otimes_1 : what we learned from the linear problem

\otimes_2 : $A = A(x,t)$, a weakly nonlinear correction to be found.

\otimes_3 : $B = B(x,t)$, a weakly dispersive correction to be found.

Plugging \otimes_2 into Φ_1 one can find to LEADING ORDER* that

KdV $\boxed{\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0}$

* dropping $\Theta(\alpha^2, \alpha\beta, \beta^2)$ terms.

— Take the Boussinesq system (dimensionless) with

$$\alpha = \beta = 0 \quad \begin{cases} \eta_t + u_x = 0 \\ \# \end{cases}$$

$$\begin{cases} \# \\ u_t + \eta_x = 0 \end{cases}$$

and lets do a DISPERSION ANALYSIS in order to get the dispersion relation: $\omega = \omega(k)$. Recall $\omega = \frac{2\pi}{T}$, $k = \frac{2\pi}{\lambda}$.

Let

$$\begin{cases} \#_1 \\ \#_2 \end{cases} \quad \begin{aligned} \eta(x,t) &= \int_{-\infty}^{\infty} \hat{\eta}_1(k) e^{i(kx - \omega(k)t)} dk + \int_{-\infty}^{\infty} \hat{\eta}_2(k) e^{i(kx + \omega(k)t)} dk \\ u(x,t) &= \int_{-\infty}^{\infty} \hat{u}_1(k) e^{i(kx - \omega(k)t)} dk + \int_{-\infty}^{\infty} \hat{u}_2(k) e^{i(kx + \omega(k)t)} dk \end{aligned}$$

Fourier amplitudes

right propag. mode

left propag. mode

The problem is linear. Therefore we can consider a mode at a time. Take a generic mode, substitute in $\#_1$ and get

$$\begin{cases} -i\omega \hat{\eta}_1 + ik \hat{\eta}_1 = 0 \\ -i\omega \hat{u}_1 + ik \hat{\eta}_1 = 0 \end{cases}$$

This system has a nontrivial solution (for the Fourier amplitudes) if $\begin{vmatrix} -i\omega & ik \\ ik & -i\omega \end{vmatrix} = 0$

We have therefore the dispersion relation

$$\omega^2 = k^2 \Rightarrow \boxed{\omega(k) = \pm k}$$

note that there have already been taken into account in $\#_2$

The system being hyperbolic we have that all Fourier modes travel with the same PHASE SPEED $C(k) = \frac{\omega}{k}$

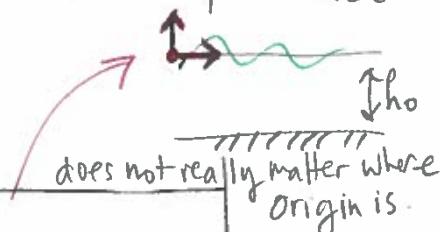
Here $C(k) = \pm 1$

$$\left\{ \begin{array}{l} \eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}_1(k) e^{ik(x-t)} dk + \int_{-\infty}^{\infty} \hat{\eta}_2(k) e^{ik(x+t)} dk \\ U(x,t) = \int_{-\infty}^{\infty} \hat{U}_1(k) e^{ik(x-t)} dk + \int_{-\infty}^{\infty} \hat{U}_2(k) e^{ik(x+t)} dk. \end{array} \right.$$

This is not the case in potential theory, where the system is **DISPERSIVE** as we will see.

How are $\hat{\eta}_1, \hat{\eta}_2, \hat{U}_1$ and \hat{U}_2 computed?

- Let's start with the linear DIMENSIONAL potential theory equations:



$$\left\{ \begin{array}{ll} \phi_{xx} + \phi_{yy} = 0 & -h_0 < y < 0 \\ \phi_{tt} + g \phi_y = 0 & y = 0 \\ \phi_y = 0 & y = -h_0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \phi_t = -g\eta \\ \eta_t = \phi_y \end{array} \right.$$

The wave elevation becomes a "passive" variable in our system since we find it after the potential problem has been solved:

$$\eta(x,t) = -\frac{1}{g} \phi_t(x,0,t)$$

— Take the Fourier representation for the potential:

$$\phi(x, y, t) = \int_{-\infty}^{\infty} \hat{\phi}_1(y_1, k) e^{i(kx - wt)} dk + \int_{-\infty}^{\infty} \hat{\phi}_2(y_1, k) e^{i(kx + wt)} dk.$$

— We want that these are harmonic functions satisfying the Neumann Condition at the bottom. Or in other words, we want to find the harmonic extension of a Fourier mode given along the (undisturbed) free surface:

$$\left\{ \begin{array}{l} \frac{d^2 \hat{\phi}_i}{dy^2} - k^2 \hat{\phi}_i = 0, \quad i=1 \& 2, \\ \frac{d \hat{\phi}_i}{dy} = 0, \quad \text{at } y = -h_0. \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \cosh(k(y+h_0)) \\ \sinh(k(y+h_0)) \end{array} \right.$$

Using the Neumann Cond:

$$\hat{\phi}_i(y, k) = F_i(k) \cosh(k(y+h_0))$$

Now we need to satisfy the "free surface" condition

$$\hat{\phi}_{yy} + g \hat{\phi}_y = 0$$

$$\rightarrow -\omega^2 F_i \cosh(kh_0) + kg F_i \sinh(kh_0) = 0$$

which leads to the DISPERSION RELATION

$$\omega^2(k) = kg \tanh(kh_0)$$

- Substituting back yields

$$\phi(x, y, t) = \int_{-\infty}^{\infty} F_1(k) \cosh(k(y+h_0)) e^{ik(x - G(k)t)} dk + \int_{-\infty}^{\infty} F_2(k) \cosh(k(y+h_0)) e^{ik(x + G(k)t)} dk$$

right going left going

where the PHASE SPEED is

$$G(k) \equiv \frac{\omega(k)}{k}, \quad c^2(k) = \frac{g}{k} \tanh(kh_0)$$

- Each Fourier mode has a phase speed that is a wave number-dependent. Long modes travel faster than shorter modes. This k -dependent phase speed characterizes a DISPERSIVE SYSTEM.

- Two limits

- SHALLOW WATERS - LONG WAVES

$$\lim_{k \rightarrow 0} G(k) = \lim_{k \rightarrow 0} \sqrt{\frac{g}{k} \tanh(kh_0)} = \boxed{\sqrt{gh_0}}. \quad \leftarrow k\text{-independent}$$

- DEEP WATERS - SHORT WAVES

$$\lim_{h_0 \rightarrow \infty} G(k) = \boxed{\sqrt{\frac{g}{k}}}$$

Dimensionless Version

$$\omega^2 = \frac{k}{\sqrt{\beta}} \tanh k\sqrt{\beta}$$

$\beta \rightarrow 0$ \bullet
 $\bullet \beta \rightarrow \infty$

— Take the system in page 76 (with ∂_x -second eqn.)

8

$$\begin{cases} \eta_t + [(1+\alpha\eta)\tilde{u}]_x - \frac{\beta}{6}\tilde{u}_{xxx} = 0 \\ \tilde{u}_t + \eta_x + \alpha\tilde{u}\tilde{u}_x - \frac{\beta}{2}\tilde{u}_{xxt} = 0 \end{cases} \xrightarrow{\alpha=0} \begin{cases} \eta_t + \tilde{u}_x - \frac{\beta}{6}\tilde{u}_{xxx} = 0 \\ \tilde{u}_t + \eta_x - \frac{\beta}{2}\tilde{u}_{xxt} = 0 \end{cases}$$

$$\Rightarrow \det \begin{bmatrix} ik - \frac{\beta}{6}(-ik^3) & -i\omega \\ -i\omega - \frac{\beta}{2}(k^2 i\omega) & ik \end{bmatrix} = 0 \quad -k^2 - \frac{\beta}{6}k^4 + \omega^2 + \omega^2 \frac{\beta}{2}k^2 = 0$$

\tilde{u} = velocity at bottom

Dispersion relation

weakly dispersive

$$\omega^2 = k^2 \frac{(1 + \cancel{\beta/6} k^2)}{1 + \beta/2 k^2} \approx (k^2 + \frac{\beta}{6} k^4) \left(1 - \frac{\beta}{2} k^2 + O(\beta^2)\right),$$

$$= k^2 + \left(\frac{\beta}{6} - \frac{\beta}{2}\right) k^4 + \dots$$

$$\omega^2 \approx k^2 - \frac{\beta}{3} k^4 + O(\beta^2).$$

"Full" dispersion relation:

$$\omega^2 = \frac{k}{\sqrt{\beta}} \tanh(k\sqrt{\beta}) \approx k^2 - \frac{\beta}{3} k^4 + O(\beta^2).$$

$$\omega^2(k) = \frac{p(k)}{q(k)} = \frac{k^2 + \beta/6 k^4}{1 + \beta/2 k^2}$$

Pade' approximation

- What can you say of a system like

$$\begin{cases} \eta_t + u_x = 0 \\ u_t + \eta_x + \frac{\beta}{3} \eta_{xxx} = 0 \end{cases}$$

$$\det \begin{bmatrix} -iw & ik \\ ik(1 - \frac{\beta}{3} k^2) & -iw \end{bmatrix} = 0 \Rightarrow \boxed{\omega^2 = k^2 \left(1 - \frac{\beta}{3} k^2\right)}$$

$k > \sqrt{3/\beta} \Rightarrow$ complex freq.

- Linear KdV (page 79)

$$\boxed{\eta_t + \eta_x + \frac{\beta}{6} \eta_{xxx} = 0}$$

$$-iw + ik + \frac{\beta}{6} (ik)^3 = 0$$

$$-iw + ik - i \frac{\beta}{6} k^3 = 0$$

$$\boxed{\omega = k - \frac{\beta}{6} k^3}$$

$$\boxed{\frac{\omega}{k} = 1 - \frac{\beta}{6} k^2}$$

- Benjamin-Bona-Mahony (BBM) '1972, Philos. Trans. Roy. Soc. London, Series A

$$(KdV) \quad \eta_t + \eta_x + \eta \eta_x + \eta_{xxx} = 0$$

$$(BBM) \quad \eta_t + \eta_x + \eta \eta_x - \eta_{xxx} = 0$$

↑ change this derivative with
the trick we saw.

- Linear BBM

$$\eta_t + \eta_x - \eta_{xxt} = 0$$

$$-i\omega + ik - (-k^2)(-i\omega) = 0$$

$$-i\omega(1+k^2) = -ik$$

$$\boxed{\omega = \frac{k}{1+k^2}} \quad \begin{array}{l} \text{BBM} \\ k \ll 1 \\ \omega \approx k(1-k^2) \end{array}$$

$$\boxed{\text{KdV} \quad \omega = k - k^3}$$

$$\boxed{\frac{\omega}{k} = \frac{1}{1+k^2}}$$

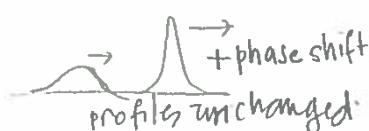
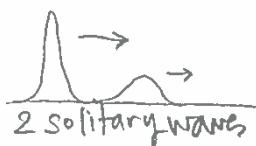
← non-physical
(ie. mathematical)
high frequencies
barely propagate.

- BBM also known as the regularized long-wave equation

↳ due to cross derivative (η_{xxt}) ↳ see 86A-B

① Facts

- Both KdV and BBM have sech^2 -solitary wave solutions (nonlinear travelling waves)
- Solitary wave ≡ pulse shaped - single hump as described by Scott Russell, 1834 which called travelling wave a wave of translation.
- KdV - called an integrable system.
 - has infinitely conserved quantities (Miura, Gardner & Kruskal '1968')
 - solitary waves are solitons - collide in an "elastic fashion"
 - profile unchanged
 - no radiation



Take

$$\frac{d\vec{y}}{dt} = A\vec{y}, \quad A \text{ diagonalizable.}$$

We may only look at

$$\frac{dy}{dt} = \lambda y, \quad y(0) = y_0,$$

and check 3 schemes:

Explicit Euler

Implicit Euler

Trapezoid Rule

$$y^{n+1} = y^n + \Delta t f(y^n, t^n)$$

$$y^{n+1} = y^n + \Delta t f(y^{n+1}, t^{n+1})$$

$$y^{n+1} = y^n + \frac{\Delta t}{2} (f^{n+1} + f^n)$$

use with \otimes

$$\text{Exp E: } y^{n+1} = y^n + \Delta t \lambda y^n = (1 + \Delta t \lambda)^{n+1} y_0 = (1+z)^{n+1} y_0 \quad z = \Delta t \lambda$$

$$\text{Imp E: } y^{n+1} = y^n + \Delta t \lambda y^{n+1}$$

$$y^{n+1} = \left(\frac{1}{1 - \Delta t \lambda}\right)^{n+1} y_0 = \left(\frac{1}{1-z}\right)^{n+1} y_0$$

$$\text{Trap: } y^{n+1} = \left(\frac{1 + \Delta t \lambda / 2}{1 - \Delta t \lambda / 2}\right)^{n+1} y_0 = \left(\frac{1 + z/2}{1 - z/2}\right)^{n+1} y_0$$

Exact solution:

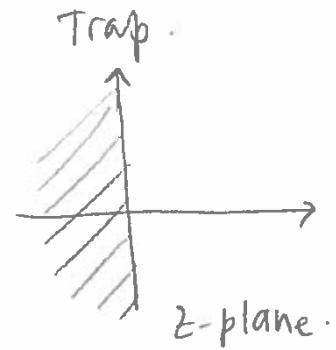
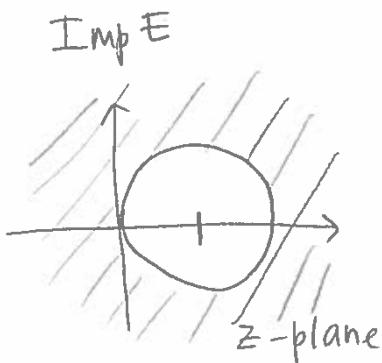
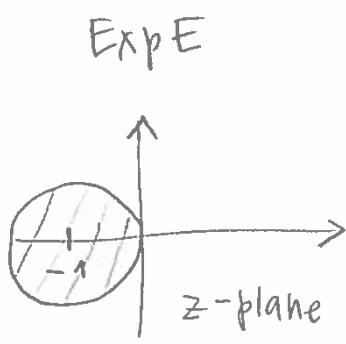
$$y^{n+1} = (e^{\Delta t \lambda})^{n+1} y_0$$

Pade' approx.

Taylor approx

When $\operatorname{Re}(\lambda) < 0$, solution should decay.

Absolute stability regions



- BBM has 3 conservation laws
- Solitary waves are NOT solitons
after interaction with other solitary waves
an oscillatory tail is generated and
the solitary wave profile changes (a bit).

- Boussinesq system (page 77)

$$\begin{cases} \eta_t + [(1+\alpha\eta)u]_x = 0 \\ u_t + \eta_x + \alpha u u_x + \frac{\beta}{3} \eta_{xx} = 0 \end{cases}$$

$$\alpha = 0$$

$$\det \begin{bmatrix} -w & k \\ k - \frac{\beta}{3}kw^2 & -w \end{bmatrix} = 0 \Rightarrow w^2 = \frac{k^2}{1 + \frac{\beta}{3}k^2}$$

$$\frac{w}{k} = \pm \sqrt{\frac{1}{1 + \frac{\beta}{3}k^2}}$$

• Facts

- Bona, Chen and Saut, J. Nonlinear Sci, Part I 2002 - linear theory
Part II 2004 - Some nonlinear theory
they study $\begin{cases} \eta_t + w_x + (w\eta)_x + aw_{xxx} - bw_{xxt} = 0 \\ wt + \eta_x + w\eta_x + cw_{xxx} - dw_{xxt} = 0 \end{cases}$
("Boussinesq abcd system")
 - ⇒ Coupled BBM
 - ⇒ Coupled KdV-BBM etc.
- provide ranges in abcd for problem to be well posed.

- A simple trick in the numerical modeling (Wei & Kirby, J. Waterw Port, Coastal, and Ocean 1995)

$$\begin{cases} \eta_t + ((1+\alpha\eta)u)_x = 0 \\ \eta_t + \eta_x + \frac{\alpha}{2}(u^2)_x - \frac{\beta}{3}u_{xxx} = 0 \end{cases}, \quad \omega^2 = \frac{k^2}{1 + \frac{\beta}{3}k^2}$$

Re-write as

$$\begin{cases} \eta_t = -((1+\alpha\eta)u)_x \\ -T_t = -\eta_x + \frac{\alpha}{2}(u^2)_x \end{cases}, \quad T = u - \frac{\beta}{3}u_{xx}$$

say use finite difference

→ get an "ODE"-system → Runge-Kutta or Predictor-Corrector

At each time step recover U^{n+1} from solving the elliptic (1-D) problem:

tridiagonal \rightarrow 

$$\frac{\beta}{3}U_{xx}^{n+1} - U^{n+1} = -V^{n+1} \quad \text{known} \quad \boxed{\text{ODE gives R.H.S.}}$$

- What can we say about

✓ (1) $u_t + uu_x = 0$ ————— Burgers

(2) $u_t + uu_x = \epsilon u_{xx}$ ————— Advection-Diffusion

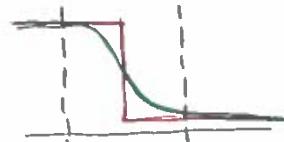
(3) $u_t + uu_x + \epsilon u_{xxx} = 0$? ————— KdV

\uparrow different ways to regularize a shock.

- (2) Ask for a traveling wave solution $u(\xi)$, $\xi = x - ct$, connecting a constant left state u^- and a constant right state u^+ . ($u^- > u^+$)

One gets that

$$u(x - ct) = \frac{u^+ + u^-}{2} - \left(\frac{u^- - u^+}{2} \right) \tanh \left(\left(\frac{u^- - u^+}{4\epsilon} \right) \xi \right)$$



Width of viscous shock wave
is proportional to (scales with): $\frac{u^- - u^+}{4\epsilon}$

- (3) KdV in standard form

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0$$

Asking for a traveling wave solution leads to an ODE that integrated yields

balance: nonlinearity \times dispersion

$$\eta(x, t) = f(x - ct) = \frac{1}{2} C \operatorname{sech}^2 \left[\frac{\sqrt{C}}{2} (x - ct - x_0) \right]$$

Can be converted back to dimensional variables.

Can not be any amplitude and width.

lets revisit the linear KdV (in a moving frame):

$$u_t + u_{xxx} = 0, \quad u(x,0) = u_0(x)$$

$$\omega = -k^3$$

Fourier solution

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_0(k) e^{i(kx - \omega(k)t)} dk$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_0(k) e^{i(kx + k^3 t)} dk$$

and this leads to the Airy function where

$$u(x,t) = \boxed{\frac{1}{(3t)^{1/3}} \int_0^{\infty} \text{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y) dy}$$

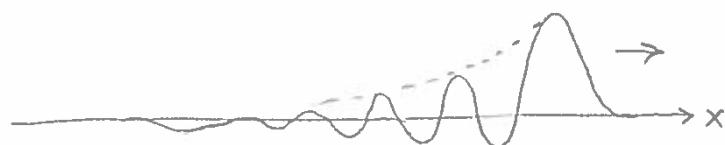
Dirac mass as $t \downarrow 0$

$$\left\{ \begin{array}{l} \text{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad \text{as } x \rightarrow \infty \\ \text{Ai}(x) \sim \frac{\cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}|x|^{1/4}} \quad \text{as } x \rightarrow -\infty \end{array} \right.$$

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx + \frac{2}{3}k^3)}$$

Where
(*)

(*) asymptotic methods for integrals

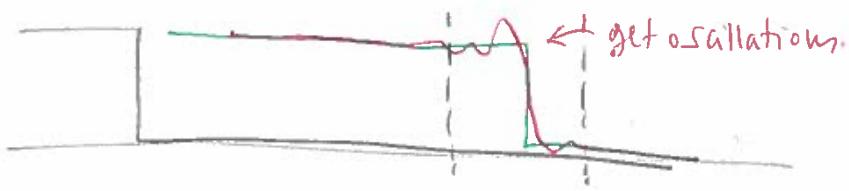




$$u_t + u_x = \varepsilon u_{xx}$$



$$u_t + u_x + \varepsilon u_{xx} = 0$$



$$u_t + \frac{1}{2} u u_x = \varepsilon u_{xx}, \text{ convergence as } \varepsilon \downarrow 0$$

$$u_t + \frac{1}{2} u u_x + \varepsilon u_{xxx} = 0, \text{ weak convergence as } \varepsilon \downarrow 0$$

- Lax and Levermore, Proc. Natl. Acad. Sci. USA, 1979

- dispersive-regularization is much harder to study.

Modified equation in numerical PDES

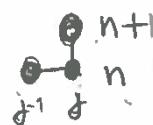
Unidirectional wave eqn

$$\frac{\partial}{\partial t} c_t + u \frac{\partial}{\partial x} c_x = 0$$

Taylor series

$$\begin{cases} c_j^{n+1} = c_j^n + \Delta t \left(\frac{\partial c}{\partial t} \right)_j^n + \frac{\Delta t^2}{2} \left(\frac{\partial^2 c}{\partial t^2} \right)_j^n + \dots \\ c_{j-1}^n = c_j^n - \Delta x \left(\frac{\partial c}{\partial x} \right)_j^n + \frac{\Delta x^2}{2} \left(\frac{\partial^2 c}{\partial x^2} \right)_j^n + \dots \end{cases}$$

UPWIND scheme molecule.



$$c_j^n = c(x_j, t^n)$$

$$\begin{aligned} \frac{\partial}{\partial t} c_t + u \frac{\partial}{\partial x} c_x |_j^n &= \left[\left(\frac{c_{j+1}^{n+1} - c_j^n}{\Delta t} \right) - \frac{\Delta t}{2} \left(\frac{\partial^2 c}{\partial t^2} \right)_j^n + O(\Delta t^2) \right] + \\ &+ u \left[\left(\frac{c_j^n - c_{j-1}^n}{\Delta x} \right) + \frac{\Delta x}{2} \left(\frac{\partial^2 c}{\partial x^2} \right)_j^n + O(\Delta x^2) \right] \end{aligned}$$

Since c is the solution to the PDE $\frac{\partial}{\partial t} c_t + u \frac{\partial}{\partial x} c_x = 0$

$$\begin{cases} \left(\frac{\partial c}{\partial t} \right)_j^n = -u \left(\frac{\partial c}{\partial x} \right)_j^n \\ \frac{\partial}{\partial t} \left(\frac{\partial c}{\partial t} \right)_j^n = -u \left(\frac{\partial}{\partial x} \left(\frac{\partial c}{\partial t} \right) \right)_j^n = -u^2 \left(\frac{\partial^2 c}{\partial x^2} \right)_j^n \end{cases}$$

$\sigma_t = 0$

Substitute above

$$\left(\frac{\partial}{\partial t} c_t + u \frac{\partial}{\partial x} c_x \right)_j^n = \left(\frac{c_{j+1}^{n+1} - c_j^n}{\Delta t} \right) - \frac{u^2 \Delta t}{2} \left(\frac{\partial^2 c}{\partial x^2} \right)_j^n +$$

$$+ u \left(\frac{c_j^n - c_{j-1}^n}{\Delta x} \right) + \frac{u \Delta x}{2} \left(\frac{\partial^2 c}{\partial x^2} \right)_j^n + O(\Delta x, \Delta t^2)$$

$\sigma = u \Delta t / \Delta x$
Courant #
 $\sigma \leq 1$.

$$\left[\left(\frac{c_{j+1}^{n+1} - c_j^n}{\Delta t} \right) + u \left(\frac{c_j^n - c_{j-1}^n}{\Delta x} \right) \right] = \left[c_t + u c_x - \frac{u \Delta x (1-\sigma)}{2} c_{xx} \right]_j^n + O(\Delta t^2, \Delta x^2)$$

$L_p = \text{discrete operator}$
(upwind)

Advection-diffusion operator

$$L_h c(x,t) = \tilde{\mathcal{L}}c + \Theta(\Delta t^2, \Delta x^2)$$

for which the discrete FDM operator is of HIGHER ORDER

MODIFIED PDE

$$\tilde{\mathcal{L}}c = 0 \Rightarrow \tilde{\mathcal{L}} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} - K \frac{\partial^2}{\partial x^2},$$

$$K \equiv \text{numerical diffusion} = \frac{u \Delta x}{2} \left(1 - \frac{u}{\Delta x / \Delta t}\right)$$

$$\tilde{\mathcal{L}}c = 0 \Rightarrow \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = K \frac{\partial^2 c}{\partial x^2}$$

- If Courant # $\sigma = 1$, to leading order NO diffusion
- If $\sigma < 1$ \Rightarrow numerical diffusion
- If $\sigma > 1$ \Rightarrow negative diffusion \Rightarrow INSTABILITY
(ill-posed problem)
- Conclusion: - backward heat equation -

original PDE local truncation error

$$L_h = \tilde{\mathcal{L}} + \text{LTE}, \quad \text{LTE} = \Theta(\Delta t, \Delta x)$$

$$L_h = \tilde{\mathcal{L}} + \tilde{\text{LTE}}, \quad \tilde{\text{LTE}} = \Theta(\Delta t^2, \Delta x^2)$$

$\tilde{\mathcal{L}}$ reveals "physical" properties of the discrete model

$$\begin{cases} c_t + u c_x = 0 \\ c(x_0) = f(x) \end{cases} \quad (\text{LeVeque, Pg 101 e 115})$$

diff

Lax-Friedrichs: $c_j^{n+1} = \frac{c_{j-1}^n + c_{j+1}^n}{2} - \frac{\sigma}{2} (c_{j+1}^n - c_{j-1}^n)$



diff

Upwind: $c_j^{n+1} = c_j^n - \sigma (c_j^n - c_{j-1}^n)$



dissip

Lax-Wendroff: $c_j^{n+1} = c_j^n - \frac{\sigma}{2} (c_{j+1}^n - c_{j-1}^n) + \frac{\sigma^2}{2} (c_{j+1}^n - 2c_j^n + c_{j-1}^n)$

dissip

Beam-Warming: $c_j^{n+1} = c_j^n - \frac{\sigma}{2} (3c_j^n - 4c_{j-1}^n + c_{j-2}^n) + \frac{\sigma^2}{2} (c_j^n - 2c_{j-1}^n + c_{j-2}^n)$



$$\sigma = \frac{u}{\Delta x / \Delta t}$$

$$L_\sigma$$

Lax-Friedrichs:

$$L_\sigma c_j^n - [c_t + u c_x]_j^n = \frac{1}{2} \left(\Delta t c_{tt} - \frac{\Delta x^2}{\Delta t} c_{xx} \right)_j^n + \dots$$

Modified equation

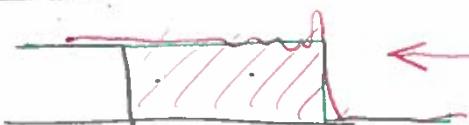
$$c_t + u c_x = \varepsilon c_{xx}, \quad \varepsilon = \frac{1}{2} \frac{\Delta x^2}{\Delta t} (1 - \sigma^2) \quad (\text{Wpg 117})$$

Lax-Wendroff:

$$L_\sigma c_j^n - [c_t + u c_x]_j^n = \frac{\Delta x^2}{6} u \left(\frac{\Delta t^2}{\Delta x^2} - \sigma^2 - 1 \right) (c_{xxx})_j^n + \dots$$

Modified equation

$$c_t + u c_x + \varepsilon c_{xxx} = 0, \quad \varepsilon = \frac{\Delta x^2}{6} u (1 - \sigma^2)$$



- PLAN - 2 problems -

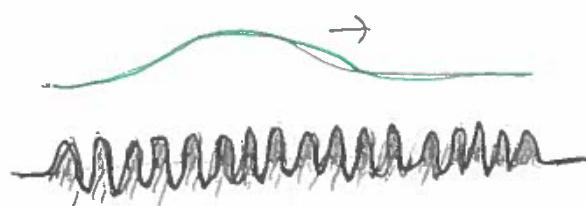
Through which we will learn new features of PDEs & Applications.



① Interface instability problem - The application

- Complex variables - Complex velocity potential
- Singular integrals
- Kelvin-Helmholtz instability and an ill-posed problem of interest
- a nonlinear evolution problem using "elliptic technique"

② Weakly nonlinear, weakly dispersive potential theory problem in the presence of a rapidly varying topography \Rightarrow rapidly varying speed



multiple scales analysis
How to obtain an effective KdV through averaging?

① Integro-differential system to be studied

$$\left\{ \begin{array}{l} \frac{\partial \bar{z}}{\partial t}(s,t) = \frac{1}{4\pi i} \int_0^{2\pi} \frac{\partial \Pi(s,t)}{\partial s} \cot \left[\frac{\bar{z}(s,t) - \bar{z}(s',t)}{2} \right] ds' \\ \frac{\partial \Pi(s,t)}{\partial t} = \frac{C}{P} K(s,t) \end{array} \right.$$

SINGULAR INTEGRAL

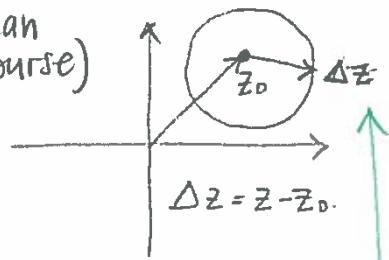
nonlinear terms

② First A review on surface tension density \rightarrow Complex Variables and Functions & ... Modeling Features of this problem.

Complex Variables REVIEW

- DERIVATIVES: PAGES 96-113 (more than needed - from another course)

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$



Limit exists \Rightarrow f is differentiable at z_0 . If $f'(z)$ exists in a neighborhood of z_0 then $f(z)$ is analytic in that neighborhood.

$$w = f(z)$$

$$f'(z) = \frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

in any direction

- Ex: $f(z) = z^2$

$$\lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z^2 + 2z\Delta z + \Delta z^2) - z^2}{\Delta z} = 2z$$

- Ex: $f(z) = \bar{z}$

$$\lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \Delta z - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta z}{\Delta z}$$

$$\begin{cases} \nearrow = 1 & \Delta y = 0 \\ \searrow = -1 & \Delta x = 0 \\ \downarrow = -i & \Delta x = \Delta y \end{cases}$$

true for $\forall z$.

nowhere differentiable!

DIFFERENTIATION FORMULAS are the same as with real valued functions. Usual rules apply:

$$\frac{d}{dz} [f(z) + g(z)] = f' + g'$$

$$\frac{d}{dz} [f(z)g(z)] = f'g + fg' \quad (\text{product rule})$$

$$\frac{d}{dz} [g(f(z))] = \underbrace{g'(f(z))}_{g'(w)} \underbrace{f'(z)}_{|_{z=f(z)}} \cdot (\text{chain rule})$$

CAUCHY - RIEMANN equations

$$\frac{df}{dz} = f'(z_0) = \lim_{z \rightarrow z_0} \frac{\Delta w}{\Delta z}$$

↑ independent of direction through which z approaches z_0 .

Therefore

if $\Delta y = 0$ (identically zero during calculation)

$$\frac{df}{dz} = \left. \frac{\partial}{\partial x} (u(x, y) + i v(x, y)) \right|_{z=z_0} = \boxed{u_x(x_0, y_0) + i v_x(x_0, y_0)}$$

if $\Delta x = 0$

$$\frac{df}{dz} = \left. \lim_{z \rightarrow z_0} \frac{\Delta w}{i \Delta y} = -i \left. \frac{\partial}{\partial y} (u + iv) \right|_{z=z_0} \right|_{\Delta z = \Delta x + i \Delta y}$$

$$= \boxed{-i (u_y(x_0, y_0) + i v_y(x_0, y_0))}$$

By equating we get a system of Partial Diff. Eqs (P)
known as the Cauchy - Riemann eqns.

short notation

$$\left\{ \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \\ \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) \end{array} \right.$$

longer notation

Complex Variables \times PDEs

① Important fact:

$f(z)$ analytic in D $(\forall z \in D)$

Then u and v are harmonic in D .

for all domain D ; open set
in

Harmonic functions satisfy Laplace's eqn. (A PDE $\Delta u = 0$)
 $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Application in (steady) temperature distributions

in Fluid Dynamics

in Electrostatic potential

etc --- to be seen

② Check: Cauchy-R.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

⊕

$$\begin{bmatrix} u_{xx} = v_{yy} \\ u_{yy} = -v_{xx} \end{bmatrix}$$

$$\begin{bmatrix} u_{xy} = v_{yx} \\ u_{yx} = -v_{xy} \end{bmatrix} \ominus$$

$$\begin{cases} \Delta u = 0 \\ \Delta v = 0 \end{cases}$$

③ Note: Continuity of 2nd-derivatives imply that order can be interchanged, namely

$$u_{xy} = u_{yx} \text{ etc...}$$

• Easy because $f(z)$ analytic \Rightarrow infinitely smooth in region of analyticity (to be verified).

Let u, v be harmonic in \mathbb{D} .

Let u, v satisfy the Cauchy-Riemann eqs.

$$u_x = v_y$$

$$u_y = -v_x$$

then v is said to be the HARMONIC CONJUGATE of u .

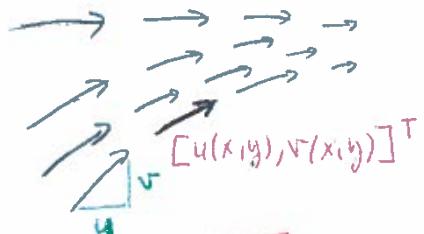
... meaning it is the imaginary part of an analytic func.
that has u as its real part.

-FLUIDS-

Let's consider HYDRODYNAMICS, namely water. In many cases we can consider the flow to be:

- (A) • incompressible
- (B) • irrotational

$\vec{u} = [u, v]^T$ is the velocity field in the fluid



(A) this can be expressed with the divergence-zero condition:

$$\operatorname{div} \vec{u} = \nabla \cdot \vec{u} = [\partial/\partial x, \partial/\partial y] [u, v]^T = [u_x + v_y] = 0$$

Recall from vector calculus:

(A1) $\nabla \cdot \vec{u}(x_0, y_0) > 0$ locally field looks like

$$\bullet \equiv (x_0, y_0)$$



(A2) $\nabla \cdot \vec{u}(x_0, y_0) < 0$



in fluids = a source or a sink of fluid.

in electrostatics — where \vec{u} expresses the electric field
then we have a positive charge density and a negative charge density

Clearly DIVERGENCE of a vector field expresses the LOCAL RATE of its divergence or convergence.

recall Divergence theorem (Gauss theorem):

$$\iint_D (\nabla \cdot \vec{u}) dA = \iint_G \vec{u} \cdot \vec{n} ds,$$

normal flux through G

(B) irrotational means curl-free:

$$\text{curl } \vec{u} = \nabla \times \vec{u} = 0$$

also called the
VORTICITY of the flow

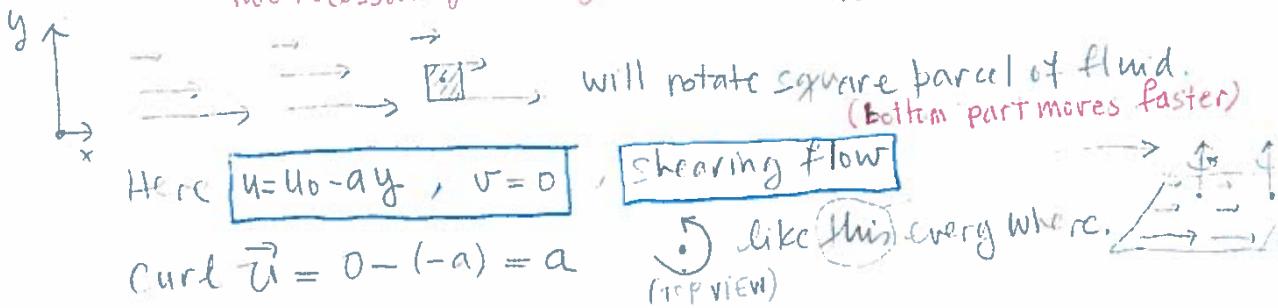
curl = expresses the local rate of rotation of
a vector field (see Stokes theorem)

For 2D vector fields

$$\text{curl } \vec{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = (v_x - u_y) \hat{k}$$

can be treated as
a scalar

- Example: has rotation.
not necessarily swirling



In summary: $\begin{cases} (A) \quad u_x + v_y = 0 \quad \text{or} \\ (B) \quad v_x - u_y = 0 \quad \text{or} \end{cases}$

$$\begin{aligned} u_x &= -v_y \\ u_y &= v_x \end{aligned}$$

This looks almost
as Cauchy-Riemann
equations

$$\begin{aligned} u_x &= (-r)y \\ u_y &= -(-r)x \end{aligned}$$

(There are different ways to do this ...)

11

As a short cut let me propose the following
Complex function, called the complex velocity potential:

$$\Phi(z) = \phi(x, y) + i\psi(x, y)$$

where ϕ is our usual velocity potential with

$$[u, v] = \nabla \phi$$

and ψ its harmonic conjugate, is called the
STREAM FUNCTION for reasons that we will see soon.

When $\Phi(z)$ is ANALYTIC, what do we get?

$$\begin{aligned} \frac{d\Phi}{dz} &= \underbrace{\phi_x}_{\partial x} + i\psi_x = -i(\psi_y + i\phi_y) = \psi_y - i\phi_y \\ \frac{d\Phi}{dz} &= u + i\psi_y = \psi_y - iv \end{aligned}$$

CR

$$\begin{aligned} \phi_x &= \psi_y \\ \phi_y &= -\psi_x \end{aligned}$$

We learn from Cauchy-Riemann equations that

(1) $\frac{d\Phi}{dz} = u - iv = \underline{\underline{\left(\frac{dz}{dt} \right)}} \equiv \text{the complex velocity}$

↑
(-1). vertical ($v = \phi_y$)
horizontal ($u = \phi_x$)

(2) $[u, v] = [\psi_y, -\psi_x] \equiv \nabla^\perp \psi, \quad \nabla^\perp \equiv (\partial y, -\partial x) \equiv \text{"grad per"}$

- (3) Level curve of the STREAMFUNCTION $\bar{\psi}$ is

$$\bar{\psi}(x, y) = \text{constant}$$

parametrize it

$$\bar{\psi}(x(s), y(s)) - \text{constant} = 0$$

then

$$\frac{d\bar{\psi}}{ds} = \left[\bar{\psi}_x \frac{dx}{ds} + \bar{\psi}_y \frac{dy}{ds} \right] = 0$$

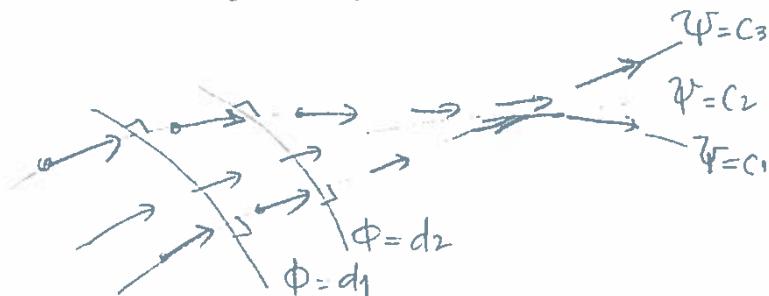
$$-v \frac{dx}{ds} + u \frac{dy}{ds} = 0$$

$$\begin{bmatrix} u, v \end{bmatrix} \begin{bmatrix} \frac{dy}{ds}, -\frac{dx}{ds} \end{bmatrix}^T = 0$$

$\underbrace{\quad}_{\vec{n} \rightarrow \bar{\psi} = \text{const}}$

Conclusion: the level curve ...

- ... $\bar{\psi}(x, y) = c$ is parallel to velocity field,
and helps us visualize the streaming
flow \Rightarrow STREAMFUNCTION name for $\bar{\psi}$.



- (4) $\nabla \phi \equiv$ normal to $\phi(x, y) = \text{constant}$
 $\nabla \bar{\psi} \equiv$ normal to $\bar{\psi}(x, y) = \text{constant}$

$$\left. \begin{aligned} \nabla \phi &= [u, v] \\ \nabla \bar{\psi} &= [-v, u] \end{aligned} \right\} \nabla \phi \perp \nabla \bar{\psi}$$

- (5) Having all this in place:

$\Phi(z) = \text{analytic}$ automatically takes into account
that the flow is irrotational and incompressible:

$$\int_G \Phi'(z) dz = 0 + i 0$$

G = any closed simple curve

↑ no circulation (swirling)

↑ no mass flux across boundary.

- (6) From (1) $\frac{d\Phi}{dz} = u - iv$ analytic

Recall:

\Rightarrow Cauchy-Riemann equations

$$\begin{cases} u_x = -v_y \\ u_y = -(-v)_x \end{cases} \Rightarrow \begin{cases} u_x + v_y = 0 & \text{(incomp.)} \\ v_x - u_y = 0 & \text{(irrotat.)} \end{cases}$$

- Take an arbitrary

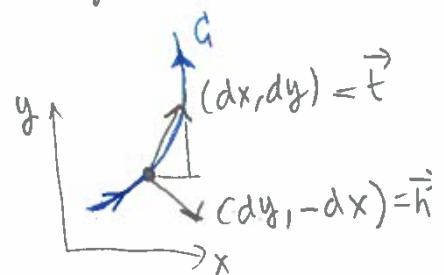
contour G :

$$\int_G \Phi'(z) dz = \int_G (u - iv)(dx + idy) = \int_G (u dx + v dy) +$$

$$+ i \int_G (v dy - u dx) = 0$$

flux through contour

circulation along contour



$\frac{d\Phi}{dz}$ is analytic + Cauchy-Goursat theorem

$$\int_G (\text{analytic}) dz = 0$$

If we want to solve a problem in a complicated domain such as D_z and we know the mapping to a simpler domain D_w , where we can find the solution more easily, then a composition such as above does the job.

Some Fluid Flow problems with Conformal Mapping:



Ex 1

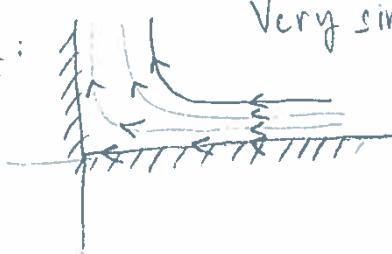
What is the complex velocity potential for the uniform horizontal flow with Speed $U > 0$?



$$\frac{d\Phi}{dz} = u - iv = U \Rightarrow \Phi(z) = Uz$$



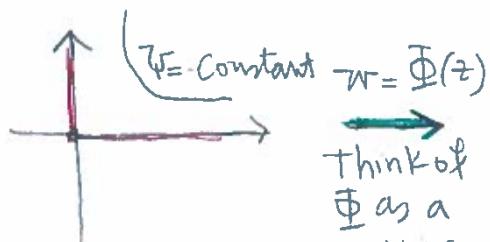
Ex 2



Very simplified model of a turning flow at a corner

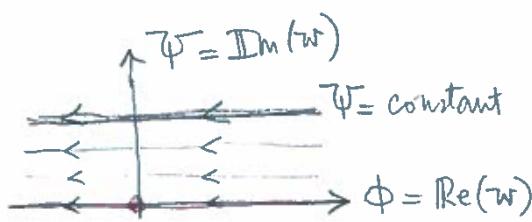
Find a complex velocity potential $\Phi(z)$.

$$w = \Phi(z) = \phi + i\psi$$



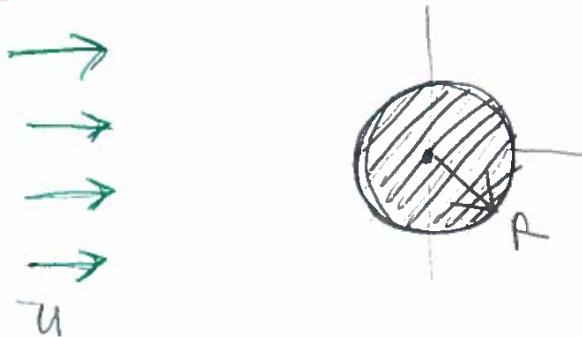
$$z = f(w)$$

We can think of (ϕ, ψ) as a coordinate system. Streamlines in the "canonical domain" (always looks the same) $\phi = \text{Const.} + \psi = \text{Const.}$





Ex 3. How about the flow around a cylinder?



Milne-Thomson Circle Theorem:

Let the free velocity potential be given by $f(z)$, analytic in the region $|z| < R$. Then in the presence of a cylinder of radius R , centered at $z=0$, the complex velocity potential is given by

$$\Phi(z) = f(z) + \overline{f\left(\frac{R^2}{\bar{z}}\right)}.$$

see page 65

$$\frac{R^2}{|z|^2} = \frac{R^2 z}{z \bar{z}}$$

Facts

(A) Along the circle ($z = R e^{i\theta}$) we have that

$$\begin{aligned} \Phi(R e^{i\theta}) &= f(R e^{i\theta}) + \overline{f\left(\frac{R^2}{R e^{i\theta}}\right)} = \\ &= f(R e^{i\theta}) + \overline{f(R e^{i\theta})} = f(z) + \overline{f(z)} \\ &= 2 \operatorname{Re} f(z) \Rightarrow \operatorname{Im} \Phi(z) = 0 \end{aligned}$$

$$\operatorname{CP}(x, y) = 0 = \text{constant}$$

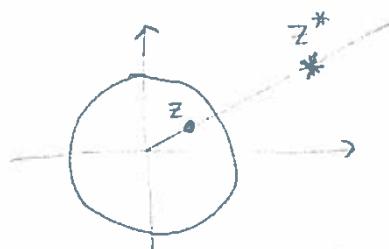
Boundary of cylinder is a streamline.

The velocity field is tangent to streamlines (level curves $\Psi(x,y) = \text{constant}$)

Therefore we conclude from the Circle theorem that the fluid goes around the cylinder.

①(B) We saw that if $f(z)$ is analytic then $\bar{f}(\bar{z})$ is analytic in the same domain D , namely for $z \in D$

①(c) The point $z^* = \frac{R^2}{\bar{z}}$ is called the symmetric point (or image point) of z with respect to the circle.



$\Rightarrow z \rightarrow 0, z^* \rightarrow \infty$
image of $z=0$ is at infinity.
image of $z=R e^{i\theta}$ "collides with itself".

Uniform flow case + a cylinder (use Circle Theorem)

$$\bullet(d) f(z) = \bar{U}z \quad \bar{f}(\bar{z}) = \bar{U}\left(\frac{R^2}{\bar{z}}\right) = \bar{U}\frac{R^2}{z}$$

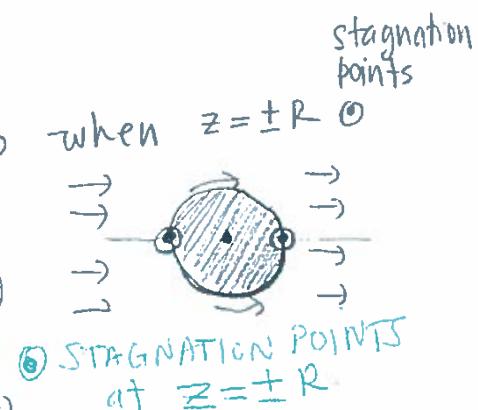
$$\Phi(z) = \bar{U}\left(z + \frac{R^2}{z}\right)$$

$$\frac{d\Phi}{dz}(z) = \bar{U}\left(1 - \frac{R^2}{z^2}\right), \quad \bar{U} - iV = 0 \quad \text{when} \quad z = \pm R$$

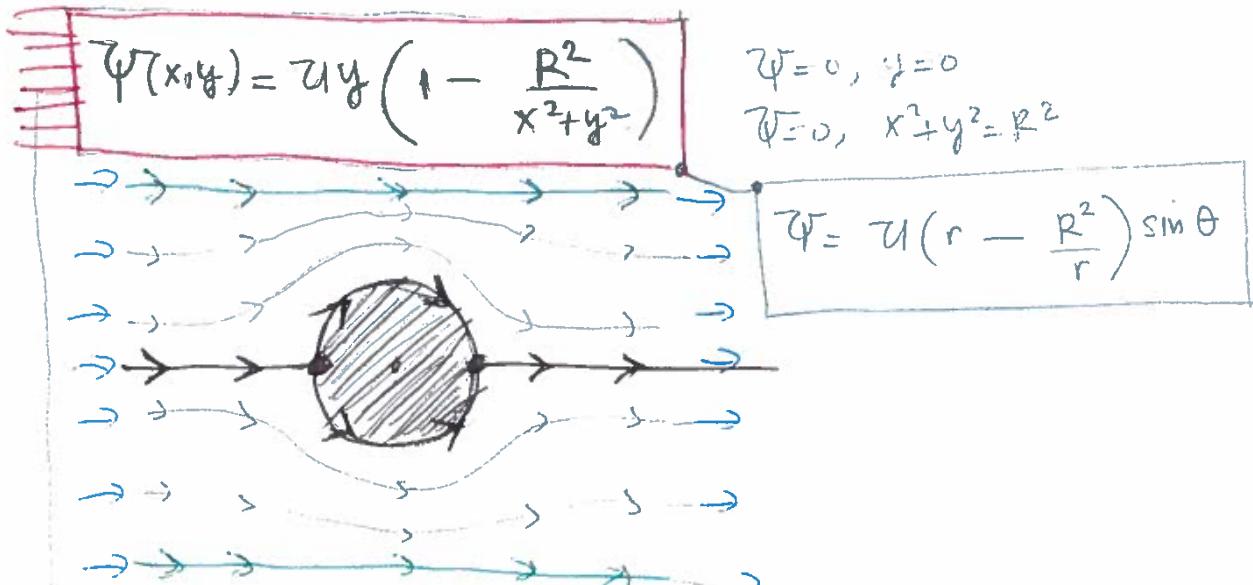
$$\int_{C_R} \frac{d\Phi}{dz}(z) dz = 0$$

not analytic

no circulation
no mass flux
no rotation
(Cx33A)



$$\begin{aligned}\Phi(z) &= U \left(z + \frac{R^2}{z} \right) = Uz + UR^2 \frac{z}{|z|^2} \\ &= \left(Ux + \frac{UR^2 x}{x^2+y^2} \right) + i \left(Uy - \frac{UR^2 y}{x^2+y^2} \right)\end{aligned}$$



Recall $U = \frac{\psi_y}{y}$ (Cx 49)

- When $|y|$ is large

$$\psi(x,y) = Uy \left(1 - \frac{R^2}{x^2+y^2} \right) \approx Uy (1-0)$$

see green streamlines

Streamline is \approx horizontal

- When $|x|$ is large

$$\psi(x,y) = Uy \left(1 - \frac{R^2}{x^2+y^2} \right) \approx Uy (1-0)$$

and flow is approximately uniform



(we saw) 2D-Laplace fund. soln.

- $\arctan y/x$ (harmonic conjugate)

Check

$$\Phi = \log y$$

source

Complex velocity potential for a point vortex

M1

strength

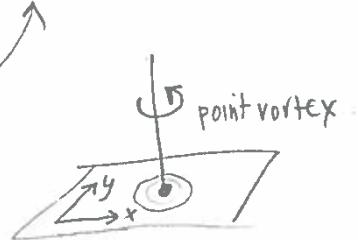
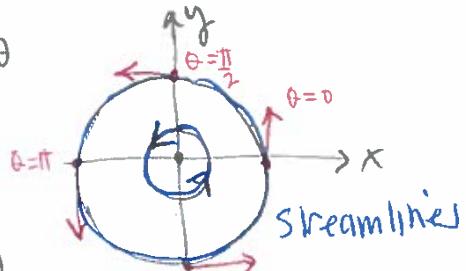
$$\Psi(z) = -ia \log(z) = -ia(\ln r + i\arg z)$$

$$\Phi(z) = a \arg z - ia \ln r = \phi(x, y) + i\psi(x, y)$$

$$\frac{d\Phi}{dz} = -ia \frac{1}{z} = -ia \frac{1}{r e^{i\theta}} = -ia e^{-i\theta}$$

$$\frac{d\Phi}{dt} = u - ir = -\frac{a}{r} (\sin \theta + i \cos \theta)$$

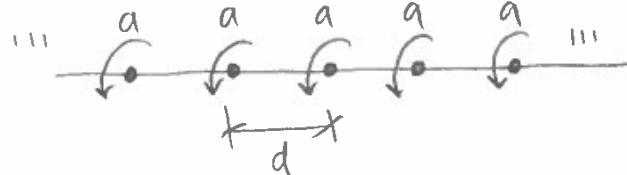
$$u = -\frac{a \sin \theta}{r}, v = \frac{a \cos \theta}{r}$$



- velocity singular at the origin

macroscopic ("far away") model for a cylindrical eddy which has collapsed to a line.

Now take



an infinite line of point vortices with a periodic structure

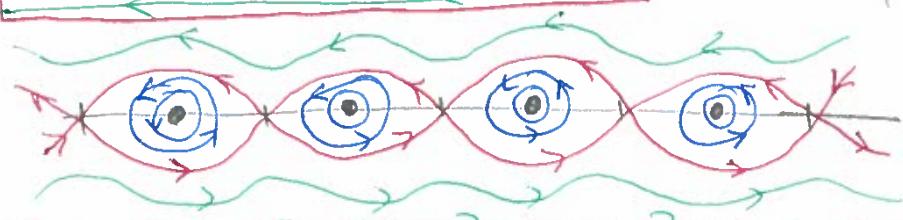
It can be shown that

M2

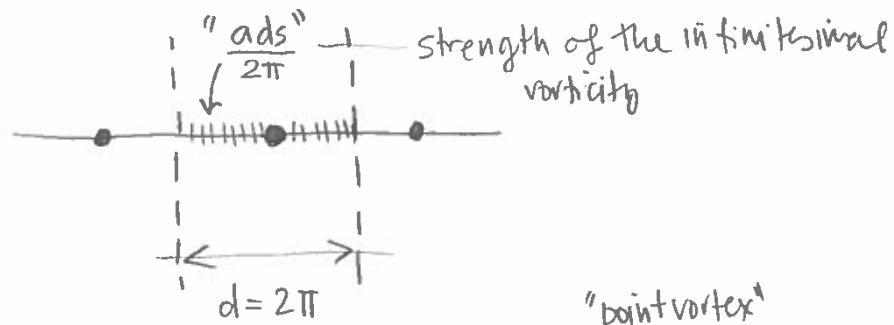
$$\Psi(z) = -ia \log \left(\sin \left(\frac{\pi z}{d} \right) \right)$$

(= far away)

Streamlines



(far away)



let

M3

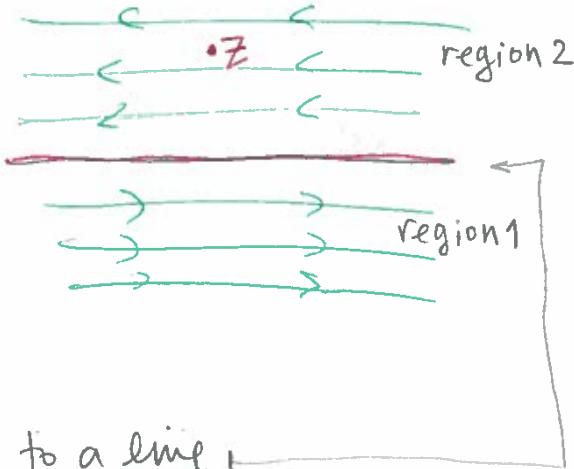
$$\Phi(z) = -\frac{i}{2\pi} \int_0^{2\pi} a \log \left(\sin \left(\frac{z-s}{2} \right) \right) ds$$

"adding" all point vortices within periodic window.

This models the flow

namely a shear flow.

Compare with figure for M2. This is like a macroscopic model for M2 where the recirculation cells collapsed to a line.



This line is a VORTEX SHEET.

$$\nabla \times \vec{u} = \nabla \times (u_i \hat{n}) = (\bar{v}_x - \bar{u}_y) \hat{k} = \omega \hat{k}, \quad \text{where } \bar{v} = 0$$

region 2: $\bar{u} \equiv \text{constant} \Rightarrow \omega = \text{vorticity} = 0$

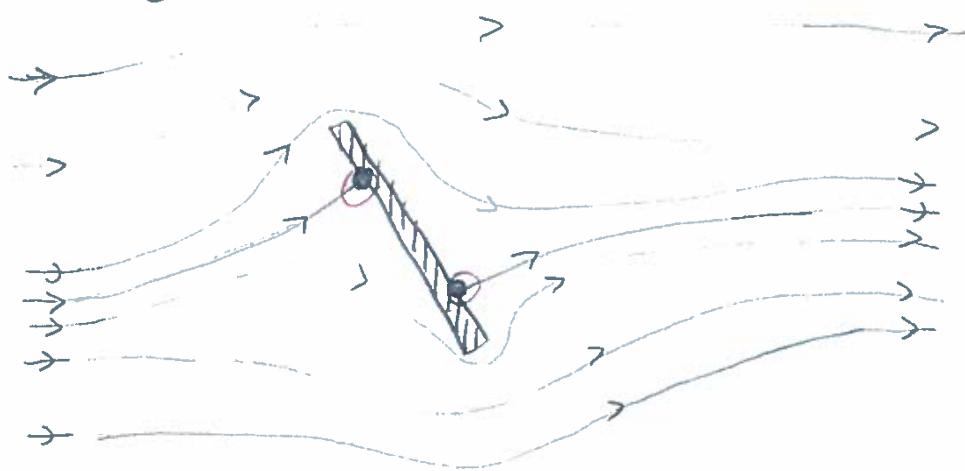
region 1: $\bar{u} \equiv \text{constant} \Rightarrow \omega = 0$

along the interface: $u(x,y) = -(u_2 + u_1) H(y)$, Heaviside.

$$\omega = -\bar{u}_y = (u_2 + u_1) \delta(y)$$

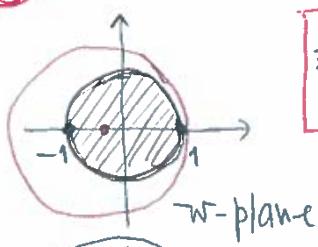
singular distribution of vorticity.

(c) putting all together for the finite length plate, including circulation and the angle of attack we can express through Complex variables a flow like

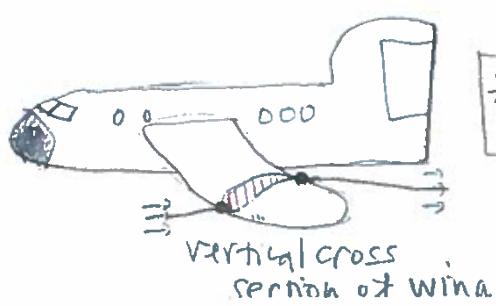
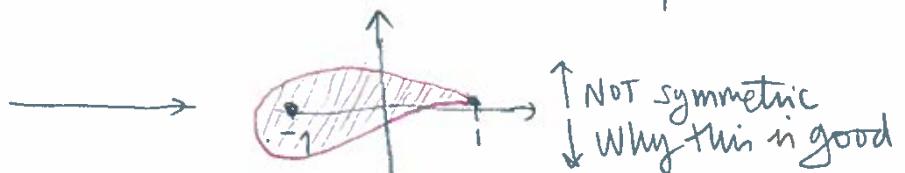
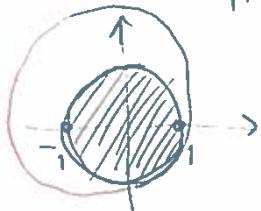
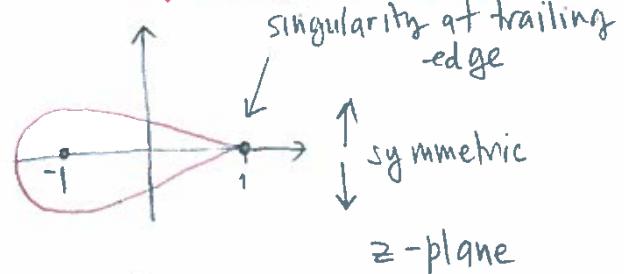


- Stagnation points, one in front, one in the back.

(d) other examples / facts: Joukowski transformation



$$z = \frac{1}{2} \left(w + \frac{1}{w} \right)$$



$$\frac{1}{2} (u^2 + v^2) + \frac{\text{pressure}}{\text{density}} = c_0$$



Consider the mapping

$$z = w + \frac{\lambda^2}{w}$$

$\lambda \in \mathbb{R}$, constant Joukowski transf

let $w = Re^{i\theta}$

$$z = Re^{i\theta} + \frac{\lambda^2}{Re^{i\theta}} = \left(R + \frac{\lambda^2}{R} \right) \cos \theta + i \left(R - \frac{\lambda^2}{R} \right) \sin \theta$$

call this x call this y

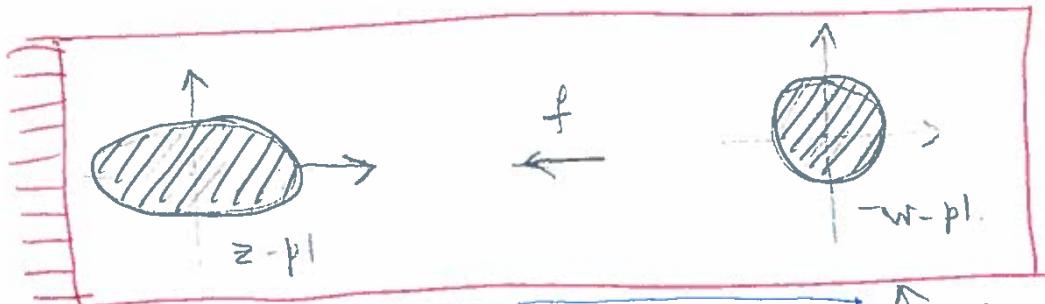
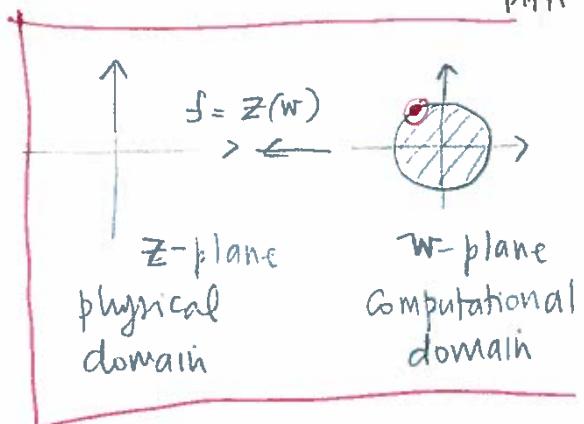
and note that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$\begin{cases} a = R + \frac{\lambda^2}{R} \\ b = R - \frac{\lambda^2}{R} \end{cases}$$

ellipse with axis equal to $2a$ and $2b$.



$$\Phi(w) = U \left(w + \frac{R^2}{w} \right) + i \frac{k}{2\pi} \log \left(\frac{w}{R} \right)$$

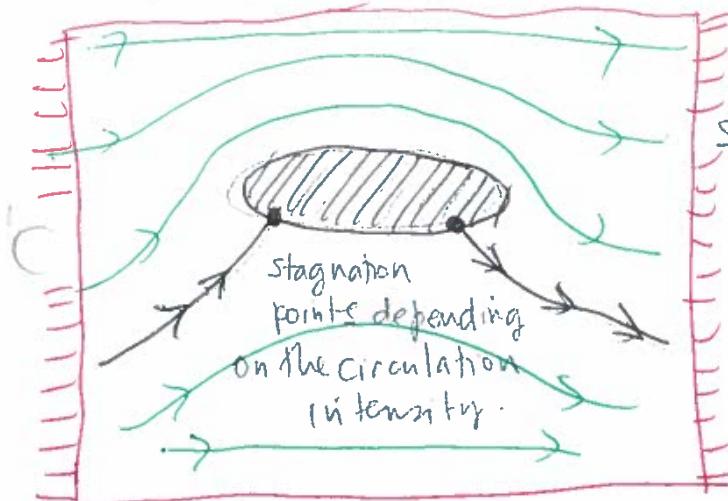
while

add circulation.

flow around cylinder of circular cross section (w -plane)

$$\Phi(z) = U \left(w(z) + \frac{R^2}{w(z)} \right) + i \frac{k}{2\pi} \log \left(\frac{w(z)}{R} \right)$$

In the z -plane is the complex velocity potential for the cylinder with elliptical cross section



The map w in

$$z = w + \frac{\lambda^2}{w}$$

Still we need to write $w = w(z)$

$$z(w) = w + \frac{\lambda^2}{w}$$

$$zw = w^2 + \lambda^2$$

$$w^2 - zw + \lambda^2 = 0$$

complex
coeff
2 root

$$w = \frac{z}{2} + \frac{\sqrt{z^2 - 4\lambda^2}}{2}$$

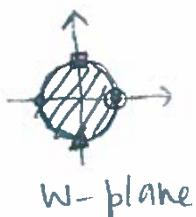
$$w(z) = \frac{z}{2} + \frac{1}{2} (z^2 - 4\lambda^2)^{1/2}$$

JOUKOWSKY transformation, an important name in classical airfoil theory in Aerodynamics

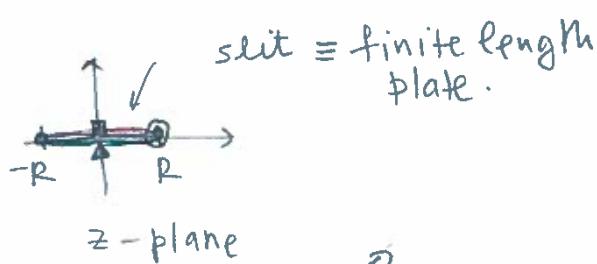
(Nikolai E. Zhukovsky 1847-1921)

① Special cases

(A) When $\lambda^2 = R$ note that one of the ellipse's axis collapses to ZERO.



$$z = w + \frac{\lambda^2}{w}$$



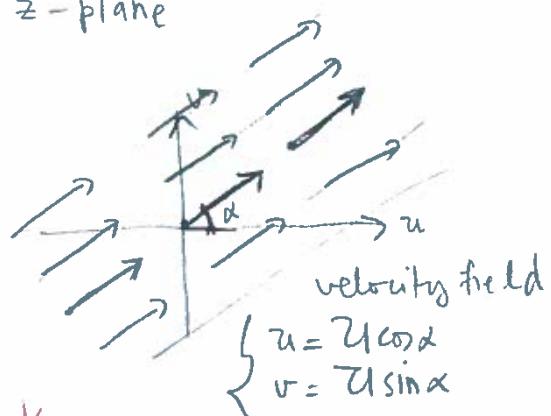
slit \equiv finite length plate.

$$(B) \Phi(z) = U e^{-iz} z$$

$$\frac{d\Phi}{dz} = U \cos \alpha - i U \sin \alpha$$

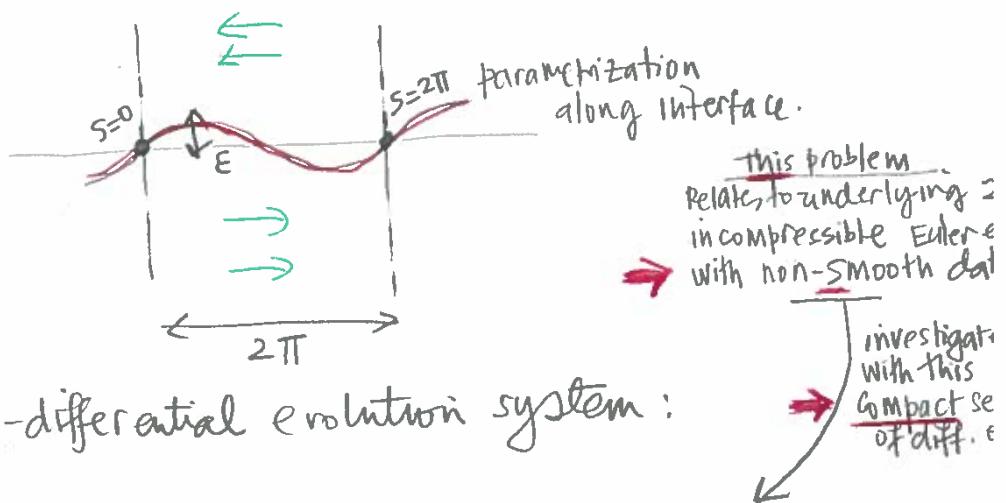
$$\left| \frac{d\Phi}{dz} \right| = U$$

$\alpha = \text{angle of attack}$



Now a hard question: is this interface STABLE to perturbations?

Let's consider a simple configuration: small periodic perturbations, namely by a single Fourier mode.



An integral-differential evolution system:

$$\begin{cases} \frac{\partial \bar{z}}{\partial t}(s_i, t) = \frac{1}{4\pi i} \int_0^{2\pi} \frac{\partial \Gamma}{\partial s}(s_i, t) \cot \left[\frac{z(s_i, t) - z(s', t)}{2} \right] ds' \\ \frac{\partial \Gamma}{\partial t}(s_i, t) = \frac{\sigma}{\rho} K(s_i, t), \quad K(s_i, t) = \frac{x_s y_{ss} - x_{ss} y_s}{(x_s^2 + y_s^2)^{3/2}} = \text{curvature} \end{cases}$$

σ = surface tension

ρ = density

Remarks:

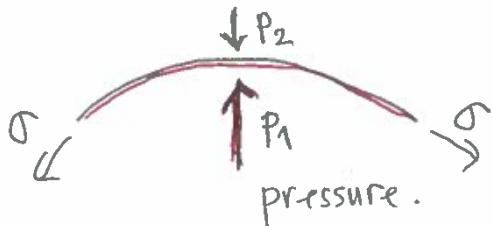
① $\frac{\partial \Gamma}{\partial s}(s_i, t)$ = vortex sheet strength that can change in time and in space (for example NONUNIFORM vortex distribution).

$\boxed{0=0} \Rightarrow \boxed{\frac{\partial \Gamma}{\partial s}(s_i, t) = a}$

② $\frac{d\bar{z}}{dz} = \bar{z} = \frac{dx}{dt} - i \frac{dy}{dt} = u - iv$

velocity of points ON the interface.

- (3) Second equation comes from Laplace-Young Law:



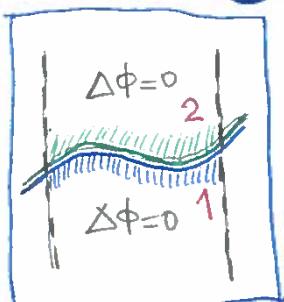
balance of stresses –
– curvature plays a role.

- (4) Last but most importantly, we have a

SINGULAR INTEGRAL, and its CAUCHY
PRINCIPAL VALUE. Let's learn how to deal
with this useful object.

- (4a) Where did it come from? It came from differentiating M_3 with respect to z and letting z approach the interface, in the flat case, approach the real axis.

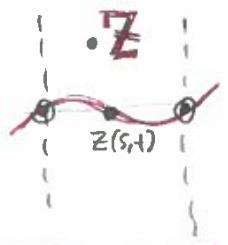
- (4b) Why useful? When z (in M_3) is away from the real axis we have an analytic function $\Phi(z)$, therefore the flow is automatically incompressible and irrotational. We automatically satisfy Laplace's equation for $\phi = \operatorname{Re}(\Phi)$.



Another way to see this is that we transformed a 2D-problem into a 1D-problem. The flow inside **DOMAIN 1** and **DOMAIN 2** are easily computed at any time.

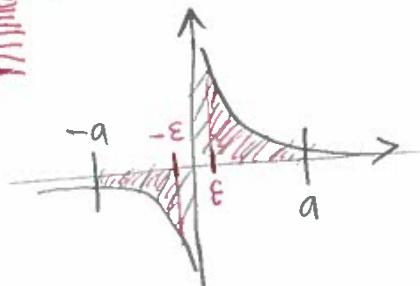
Say we use M3 in the form

$$\bullet \quad \text{Res}(z) = -\frac{i}{2\pi} \int_0^{2\pi} \frac{\partial P(s,t)}{\partial s} \cot \left[\frac{z - z(s,t)}{2} \right] ds.$$



4c SINGULAR INTEGRAL

$$\int_{-a}^a \frac{1}{x} dx = ?$$



Cauchy Principal Value

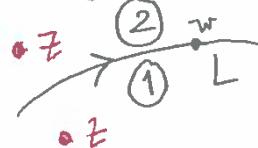
$$\text{PV} \int_{-a}^a \frac{1}{x} dx = \int_{-a}^a \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0}$$

$$+ \int_{-\epsilon}^{\epsilon} \frac{1}{x} dx = 0 \quad \text{in this case.}$$

(4d)

Consider the Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(w)}{w-z} dw$$



Ablowitz & Fokas
Complex Variables
page 517, CUP.

where L is a smooth curve (arc or closed contour).

Let the boundary data $\varphi(w)$ be Hölder continuous:

$$|\varphi(w_1) - \varphi(w_2)| \leq G |w_1 - w_2|^\alpha, \quad 0 < \alpha \leq 1$$

(Note $\alpha=1 \Rightarrow$ Lipschitz condition)

Note $F(z)$ is analytic provided $z \notin L$.

(in Salsa there is a real-valued (\mathbb{R}^n) version - page 145)

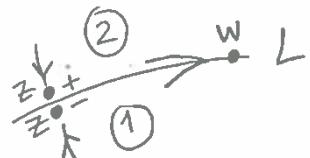
11:

- **PLEMELJ formula** (Sokhotski-Plemelj theorem)
(1868) (1908)

for the limiting values of $F(z)$ as z approaches L :

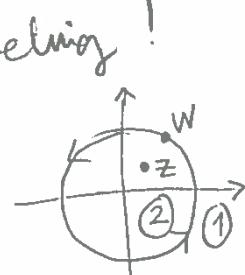


$$F(z) = \pm \frac{1}{2} \varphi(z) + \frac{1}{2\pi i} \int_L \frac{\varphi(w)}{w-z} dw$$



- Therefore note that for $z \in L$ the value of $\boxed{\quad}$ is ambiguous.
- The function $F(z)$ is said to be sectionally analytic:
 - Analytic in domain ②
 - Analytic in domain ①
 - Jumps across curve separating domains
- Very convenient object for our modeling!

Example: What is $\frac{1}{2\pi i} \int_C \frac{1}{w-z} dw$?
 (application)



We know easily that when

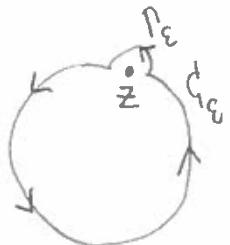
$$z \in \textcircled{2} \quad \frac{1}{2\pi i} \int_C \frac{1}{w-z} dw = 1 \quad (\text{winding number of } C)$$

$$z \in \textcircled{1} \quad \frac{1}{2\pi i} \int_C \frac{1}{w-z} dw = 0$$



Plemelj:
$$\left. \begin{aligned} & \textcircled{2} \quad \frac{1}{2} + \frac{1}{2\pi i} \int_C \frac{1}{w-z} dw = 1 \\ & z \rightarrow C \quad \textcircled{1} \quad -\frac{1}{2} + \frac{1}{2\pi i} \int_C \frac{1}{w-z} dw = 0 \end{aligned} \right\} \quad \left. \begin{aligned} & \frac{1}{2\pi i} \int_C \frac{1}{w-z} dw = \frac{1}{2} \end{aligned} \right\}$$

$$\frac{1}{2\pi i} \int_G \frac{1}{w-z} dw = \text{Cauchy Theorem}$$



$$\frac{1}{2\pi i} \int_G \frac{1}{w-z} dw = \frac{1}{2\pi i} \int_{G_\epsilon} \frac{1}{w-z} dw + \frac{1}{2} = \epsilon \downarrow 0 = \dots$$

"1/2 residue"

$$\dots = \frac{1}{2\pi i} \int_G \frac{1}{w-z} dz + \frac{1}{2} = 1 \text{ (winding \#)}$$

$\underbrace{\quad}_{= 1/2}$



$$\frac{1}{2\pi i} \int_G \frac{1}{w-z} dw = \frac{1}{2\pi i} \int_G \frac{1}{w-z} dz - \frac{1}{2} = 0 \text{ (winding \#)}$$

$\underbrace{\quad}_{= 1/2}$

D. Moore - the spontaneous appearance of a singularity
in the shape of an evolving vortex sheet.

Proc. R. Soc. London A., 1979

Studied

$$\left\{ \begin{array}{l} \frac{\partial z}{\partial t}(r, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dr'}{z(r, t) - z(r', t)} \\ z(r, 0) = r + i\varepsilon \sin r \end{array} \right.$$

\uparrow smooth analytic initial profile.

uniform vortex sheet strength: $a=1$.

(a) linear problem known to be ill-posed.

(b) will nonlinearity restore well-posedness, through say
the coupling of Fourier modes?

(c) Moore used an asymptotic analysis to conjecture
(not a proof) that

- the critical time is proportional to $\ln(1/\varepsilon)$

- Fourier coefficients decay like $k^{-2.5}$

→ Burgers $t_c = O(1/\varepsilon)$

→ second derivative should blow-up.

(d) "replacing a vortex layer by a vortex sheet is
very restricted" — A modeling issue?
How to fix the model?

of Mathematical (Diff. Eq + Singular Integral)

Lloyd Treffethen

Spectral Methods in MATLAB
Book by SIAM

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(a) can be handled by standard methods of analysis, and part (b) is a corollary. Parts (c) and (d) are known as Paley-Wiener theorems; see [Krat76] and [PaWe34].

Theorem 1 Smoothness of a function and decay of its Fourier transform.

Let $u \in L^2(\mathbb{R})$ have Fourier transform \hat{u} .

* (a) If u has $p-1$ continuous derivatives in $L^2(\mathbb{R})$ for some $p \geq 0$ and a p th derivative of bounded variation,* then

$$\hat{u}(k) = O(|k|^{-p-1}) \quad \text{as } |k| \rightarrow \infty.$$

(b) If u has infinitely many continuous derivatives in $L^2(\mathbb{R})$, then

$$\hat{u}(k) = O(|k|^{-m}) \quad \text{as } |k| \rightarrow \infty$$

for every $m \geq 0$. The converse also holds.

(c) If there exist $a, c > 0$ such that u can be extended to an analytic function in the complex strip $|\operatorname{Im} z| < a$ with $\|u(\cdot + iy)\| \leq c$ uniformly for all $y \in (-a, a)$, where $\|u(\cdot + iy)\|$ is the L^2 norm along the horizontal line $\operatorname{Im} z = y$, then $\hat{u}_a(k) \in L^2(\mathbb{R})$, where $\hat{u}_a(k) = e^{iky}\hat{u}(k)$. The converse also holds.

(d) If u can be extended to an entire function (i.e., analytic throughout the complex plane) and there exists $a > 0$ such that $|u(z)| = o(e^{|z|})$ as $|z| \rightarrow \infty$ for all complex values $z \in \mathbb{C}$, then \hat{u} has compact support contained in $[-a, a]$; that is,

$$\hat{u}(k) = 0 \quad \text{for all } |k| > a.$$

The converse also holds.

This theorem, although technical, can be illustrated by examples of functions that satisfy the various smoothness conditions.

Illustration of Theorem 1(a). Consider the step function $s(x)$ defined by

$$s(x) = \begin{cases} \frac{1}{2}, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

*A function f has bounded variation if it belongs to $L^1(\mathbb{R})$ and the supremum of $\int f g'$ over all $g \in C^1(\mathbb{R})$ with $|g(x)| \leq 1$ is finite [Zie89]. If f is continuous, this coincides with the condition that the supremum of $\sum_{j=1}^N |f(x_j) - f(x_{j-1})|$ over all finite samples $x_0 < x_1 < \dots < x_N$ is bounded.

* Moore! $p = 1, 5 \Rightarrow$ does not have a 2nd derivative.



Fig. 4.1. A step function s is used to generate piecewise polynomials—B-splines—by convolution.

This function is not differentiable, but it has bounded variation. To generate functions with finite numbers of continuous derivatives, we can take convolutions of s with itself. The convolution of two functions u and v is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(y)v(x-y) dy = \int_{-\infty}^{\infty} v(y)u(x-y) dy, \quad (4.1)$$

and the functions s , $s * s$, and $s * s * s$ are sketched in Figure 4.1. These functions, known as B-splines, are piecewise polynomials that vanish outside the intervals $[-1, 1]$, $[-2, 2]$, and $[-3, 3]$, respectively [Boo78].

The function s is piecewise constant and satisfies the condition of Theorem 1(a) with $p = 0$. Similarly, $s * s$ is piecewise linear, satisfying the condition with $p = 1$, and $s * s * s$ is piecewise quadratic, satisfying it with $p = 2$. Now the Fourier transforms of these functions are

$$\hat{s}(k) = \frac{\sin k}{k}, \quad \widehat{s * s}(k) = \left(\frac{\sin k}{k} \right)^2, \quad \widehat{s * s * s}(k) = \left(\frac{\sin k}{k} \right)^3.$$

These results follow from the general formula for the Fourier transform of a convolution,

$$\widehat{u * v}(k) = \hat{u}(k)\hat{v}(k), \quad k \in \mathbb{R}. \quad (4.2)$$

Just as predicted by Theorem 1(a), these three Fourier transforms decay at the rates $O(|k|^{-1})$, $O(|k|^{-2})$, and $O(|k|^{-3})$.

Illustration of Theorem 1(d). By reversing the roles of \hat{s} , $\widehat{s * s}$, $\widehat{s * s * s}$ and s , $s * s$, $s * s * s$ in the above example, that is, by regarding the former as the functions and the latter as the transforms (apart from some unimportant constant factors), we obtain illustrations of Theorem 1(d). The function \hat{s} , for example, satisfies $\hat{s}(k) = o(e^{|k|})$ as $|k| \rightarrow \infty$, and its Fourier transform, $2\pi\hat{s}(x)$, has compact support $[-1, 1]$.

Illustration of Theorem 1(c). Consider the pair

$$u(x) = \frac{\sigma}{x^2 + \sigma^2}, \quad \hat{u}(k) = \pi e^{-\sigma|k|} \quad (4.3)$$

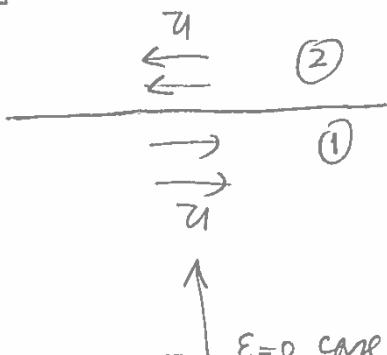
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Stability analysis of

From Plemelj:

$$\left. \begin{array}{l} \frac{\partial \bar{z}}{\partial t}(s, t) = \frac{1}{4\pi i} \int_0^{2\pi} \frac{\partial \Gamma(s', t)}{\partial s} \cot\left(\frac{z-z'}{2}\right) ds' \\ \frac{\partial \Gamma}{\partial t}(s, t) = \frac{1}{\rho} \left[\frac{x_s y_{ss} - x_{ss} y_s}{(x_s^2 + y_s^2)^{3/2}} \right] \end{array} \right\} \quad \left(\begin{array}{l} \text{average speed} \\ \frac{u^2 + U^0}{2} \end{array} \right),$$

Underlying stationary solution



Write

$$\left. \begin{array}{l} \Gamma(s, t) = 2u(s + \varepsilon \tilde{\Gamma}(s, t)) \\ x(s, t) = s + \varepsilon \tilde{x}(s, t) \\ y(s, t) = \varepsilon \tilde{y}(s, t) \end{array} \right\} \quad \xrightarrow{\varepsilon=0 \text{ case}}$$

Plug Φ_2 into Φ_1 to get the $\Theta(\varepsilon)$ -equations

$$\left. \begin{array}{l} \frac{\partial \tilde{\Gamma}}{\partial t} = \frac{1}{2u\rho} \tilde{y}_{ss} \\ \frac{\partial \tilde{x}}{\partial t} = -\frac{u}{4\pi} \int_0^{2\pi} \frac{\tilde{y}(s, t) - \tilde{y}(s', t)}{\sin^2\left(\frac{s-s'}{2}\right)} ds' \\ \frac{\partial \tilde{y}}{\partial t} = -\frac{u}{4\pi} \left[\int_0^{2\pi} \frac{\tilde{x}(s, t) - \tilde{x}(s', t)}{\sin^2\left(\frac{s-s'}{2}\right)} ds' - \int_0^{2\pi} 2\tilde{\Gamma}_s(s', t) \cot\left(\frac{s-s'}{2}\right) ds' \right] \end{array} \right\}$$

\square_{2p} \square_{1p}

Still seems like a very complicated system, until one recognizes the singular integrals as Hilbert transforms on the circle.

Hilbert transform

$$\square_1 \quad \mathcal{H}_6[f](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

$$\square_2 \quad \mathcal{H}_6[f'](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{(x-y)^2} dy$$

$$\widehat{\mathcal{H}_6 f}(k) = \underbrace{-i \operatorname{sign}(k) \widehat{f}(k)}_{\text{Fourier symbol or multiplier}} \quad \text{Fourier transform}$$

\square_{1p} is the periodic version of \square_1

\square_{2p} is the periodic version of \square_2 .

We also have that

$$\square_{1p} \quad \frac{1}{2\pi} \int_0^{2\pi} \sin(ks') \cot\left(\frac{s-s'}{2}\right) ds' = -\cos ks$$

$$\square_{2p} \quad \frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(ks) - \cos(ks')}{\sin\left(\frac{s-s'}{2}\right)} ds' = k \cos(ks)$$

Let's do the Fourier analysis of ϕ_3 with

$$\left. \begin{array}{l} \tilde{x}(s,t) = A_1(t) \cos ks \\ \tilde{x}(s,t) = A_2(t) \cos ks \\ \tilde{y}(s,t) = A_3(t) \cos ks \end{array} \right\} \quad \phi_4$$

Note that \sin and \cos decouple in ϕ_3 .

Substitute ϕ_4 in ϕ_3 to get (using \square_{1p} & \square_{2p})

$$\frac{d}{dt} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k^2 \sigma / (2\mu p) \\ 0 & 0 & -\mu k \\ \mu k & -\mu k & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$\lambda(k) = \pm \left[\mu^2 k^2 - \frac{\sigma}{2p} k^3 \right]^{1/2}, \quad \lambda(k) = 0$$

→ go to page 121-125 and back.

$\sigma=0 \Rightarrow$ problem is ill-posed. High wave numbers grow in an unbounded fashion

— Another way to see things. Take the solution of linear prob. as a Fourier series.

$$z(s,0) = s + \sum \hat{z}_k e^{iks}, \quad \text{where}$$

$$\hat{z}_k \sim C e^{-\beta k} \quad (\text{analytic initial data according to } \# \text{page 117-})$$

— Solution has a component of the form

$$z(s,t) = s + \sum \underbrace{\hat{z}_k e^{1+\lambda(k)t} e^{iks}}_{\text{asympt.}} e^{-K(\beta-\mu t)}$$

↑ at a critical time
no more decay!
Fourier series makes
NO SENSE.

D. Ambrose, SIAM J. Math Analysis, 2003 → Well-posedness of vortex sheet w/ surface tension

INTERFACE INSTABILITY PROBLEM

EVOLUTION of PERTURBED INTERFACE : (J.F.M. 1986) , R.Krasny

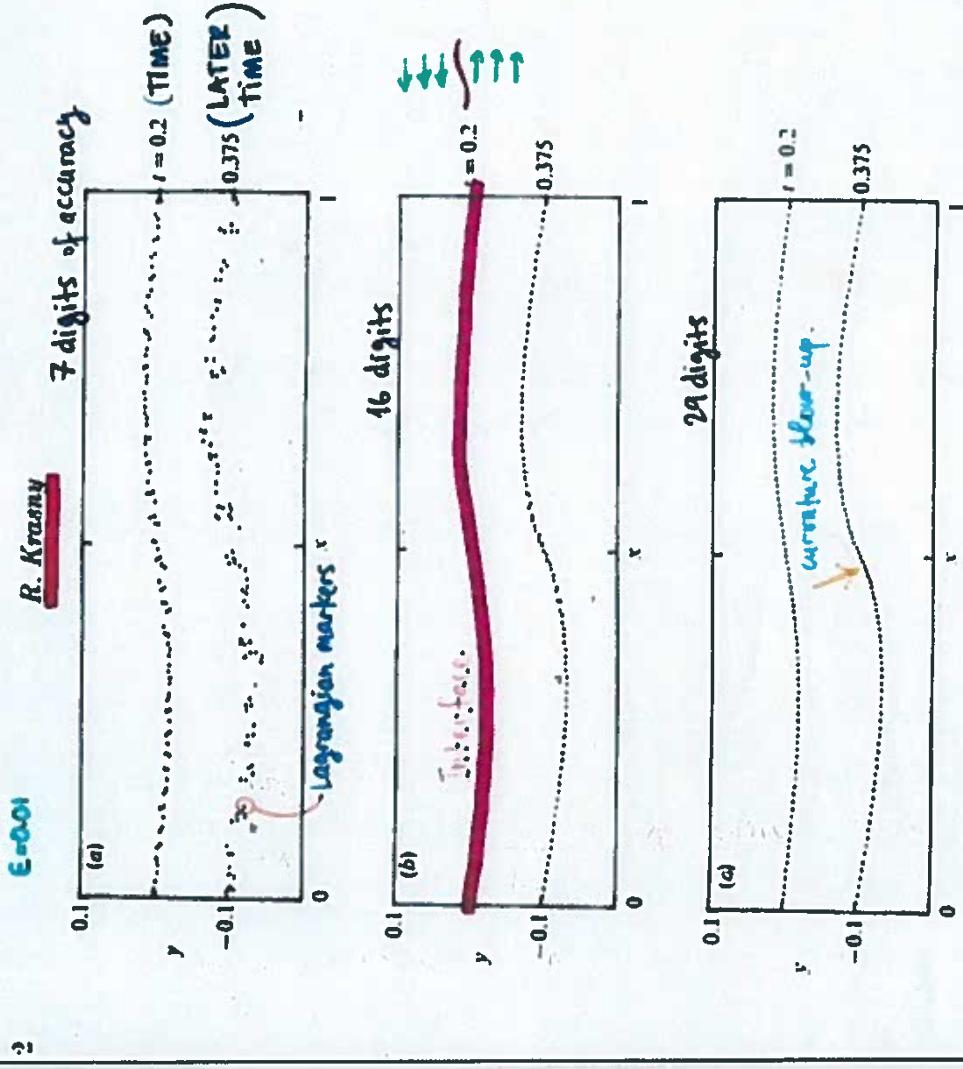


FIG. 3. Numerical solution of the point-vortex equations (1.5) with initial condition (2.1) using fifth-order Runge-Kutta integration ($N = 100$, $\Delta t = 0.01$ for $t \leq 0.25$ and $\Delta t = 0.001$ for $t > 0.25$).
 (a) single precision (7 digits); (b) double precision (16 digits); (c) CD 'double precision (29 digits).
 The solution is plotted at $t = 0.2$, 0.375 .

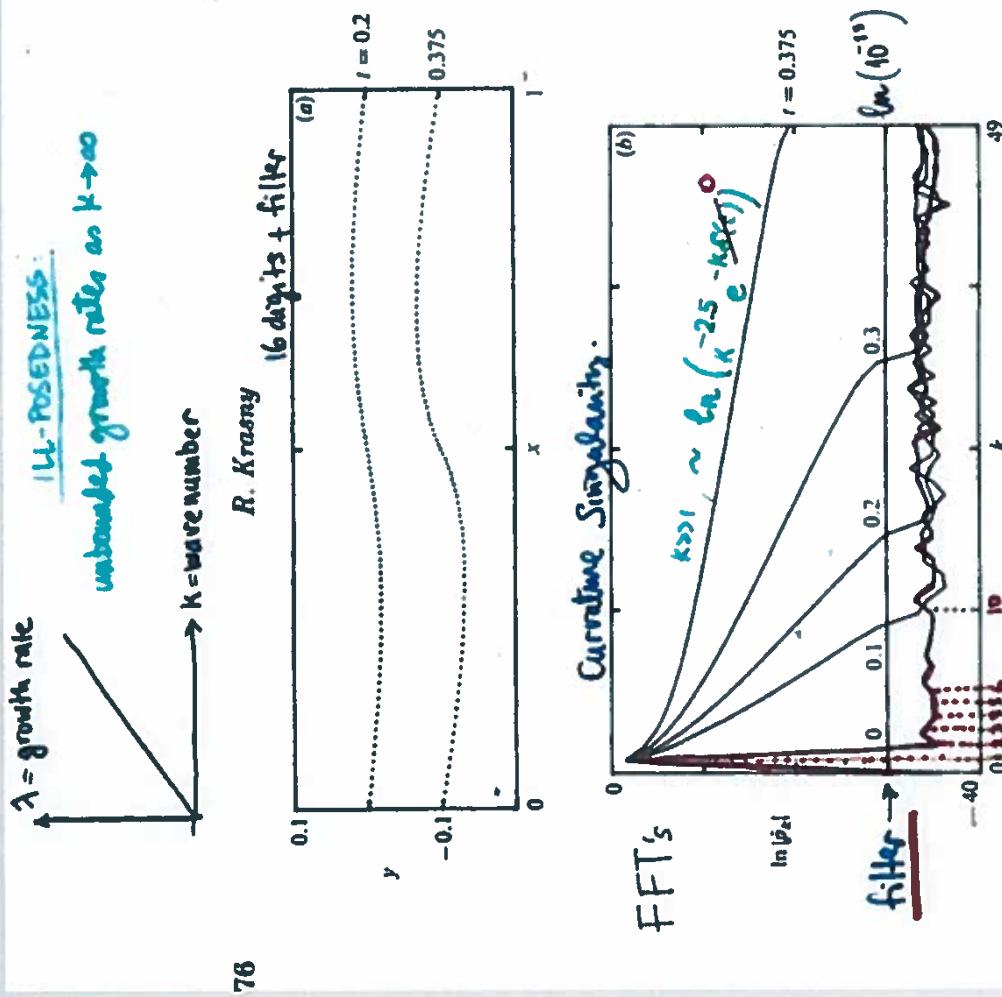


FIGURE 5. Double-precision (16 digit) calculation with the filter level set at 10^{-10} (the horizontal line in (b)). This calculation used $N = 100$ and the same time stepping as before. (a) point-vortex positions; (b) log-linear plot of the Fourier coefficients' (2.2) amplitudes versus wavenumber. Compare this with the unfiltered and the higher precision calculations in figures 3 and 4.

filter prevents roundoff error from generating spurious irregular motion of markers.

VORTEX
PILOT
METHOD

$$\left\{ \begin{array}{l} \frac{\partial \xi}{\partial t} = \frac{1}{2} \int_0^1 \frac{\sinh \pi i(y-\tilde{y})}{\cosh^2(y-\tilde{y}) - \cosh 2\pi(x-\tilde{x}) + \delta_{\text{eff}}^2} d\tilde{y} \\ \frac{\partial \eta}{\partial t} = \frac{1}{2} \int_0^1 \frac{\sin 2\pi(x-\tilde{x})}{\cosh^2(y-\tilde{y}) - \cosh 2\pi(x-\tilde{x}) + \delta_{\text{eff}}^2} d\tilde{y} \end{array} \right.$$

PULL-UP of the INTERFACE (Prasang)
J.C.P. 1986

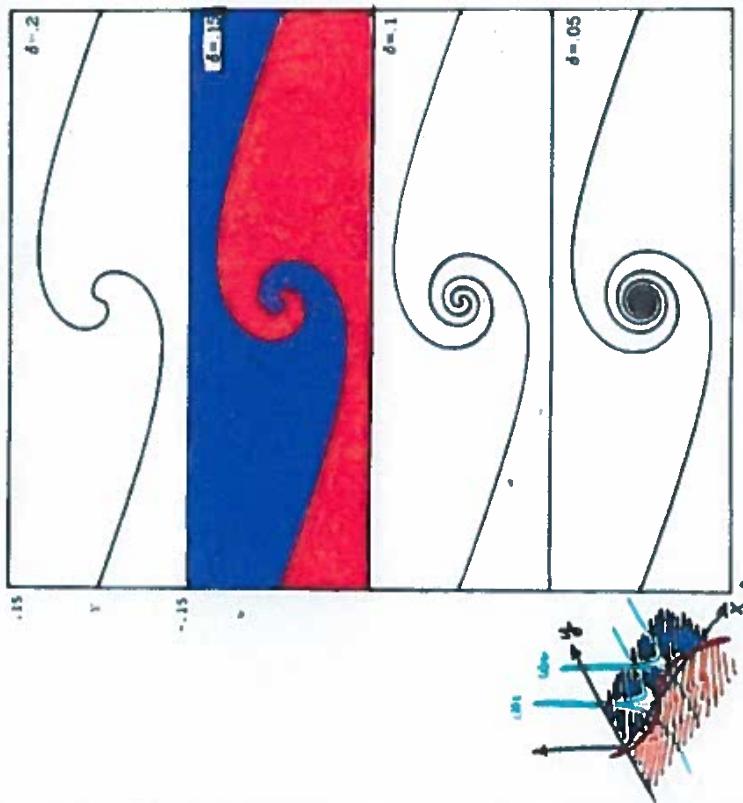


FIG. 5. Solution of the δ equations (1), (2) at $t = 10$ using $\delta = 0.2, 0.15, 0.05$.

δ = desingularization parameter (singular integrals)



Triggiani et al.
NANIEK STROES



Fig. 1 Computed roll-up of the material interface for each of the cases shown in Figure 1.

be clearly seen. Indeed, if the strain rate experienced by these layers coincides with the total circulation of the vortex, then the resulting equilibrium strain-limited vorticity diffusion layer thickness should be expected to scale like $\delta \sim Re^{-1/2}$. Note that this is at least qualitative agreement with the results shown. For the purposes of our discussion here of the laminar behavior, we note that this would suggest that the thickness δ of these layers will tend to zero as $Re \rightarrow \infty$ regardless of the initial thickness δ_0 .

Figure 2 shows the position at $t = 2$ of the material interface for each of the cases shown in Fig. 1. For the lowest Reynolds number and the largest δ (top left frame), relatively little roll-up has taken place, but increasing δ or reducing δ causes this transition to stretch more and wrap up more around the vortex center. The larger number of vortices corresponds to the smaller initial layer thickness and the largest Reynolds number (lower right frame). Comparing Figs. 1 and 2 shows that, for the lowest Re , it is not possible to identify the position of this material interface from the vorticity distribution at this relatively early stage of the layer evolution. However, for the higher Reynolds numbers, the vorticity distribution more clearly corresponds to the material interface for at least the outermost vortices of the vortex. Indeed, since at these higher Re 's the strain-diffusion balance produces a comparatively small vorticity layer thickness at and past this interface, the layers corresponding to the outer few vortices remain relatively well separated from adjacent ones.

In Figs. 3 and 4 we show results from calculations of the roll-up process using the regularized inviscid vortex sheet model. The interface itself is shown in Fig. 4 and the vorticity, as constrained by equation (4), is shown in Fig. 3. In these inviscid calculations, the regularization scale δ is the only parameter to vary. These results show that, as this regularization scale is reduced, similar development as seen in Figs. 1 and 2 is observed. Smaller regularization tends to increased roll-up in the core, but the shape of the interface away from the core is relatively independent of δ . While there are considerable similarities between the results from these viscous and inviscid calculations, it is clear that for a given value of δ there is much more roll-up for the vortex sheet model than for the full vorticity calculation with the same initial layer thickness and vorticity distribution. A comparison of the vorticity distributions in Figs. 1 and 3 amplifies these differences. In particular, for the vortex sheet calculations

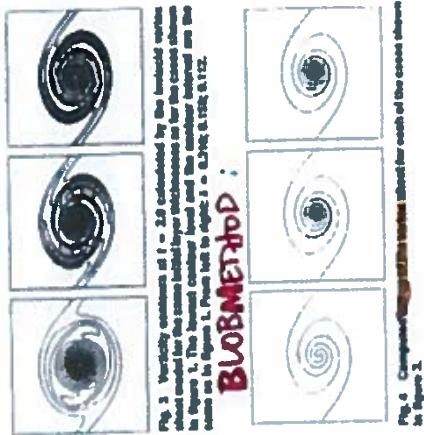


Fig. 2 Computed roll-up of the material interface for each of the cases shown in Figure 1.

N-S: PTS. of MAX VORTICITY

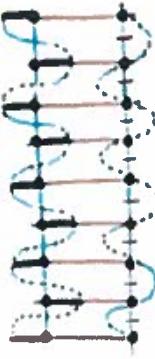
$$\begin{array}{c} \leftarrow \leftarrow \leftarrow -U \\ \hline \leftarrow \leftarrow \leftarrow +U \\ \leftarrow \leftarrow \leftarrow \end{array}$$

$$\Gamma_j(t) = 2Ue_j, \quad X_j(t) = e_j + \varepsilon y_j(t) = 0$$

$$\left\{ \begin{array}{l} \Gamma_j(t) = 2U(e_j + \varepsilon Y_j^\varepsilon(t)) \\ X_j(t) = e_j + \varepsilon Y_j^\varepsilon(t) \\ y_j(t) = \varepsilon Y_j^\varepsilon(t) \end{array} \right.$$

Neglect

$$\cos(N/2\varepsilon)$$



$$\sin(N/2\varepsilon)$$



$$D = \text{derivative}$$

$$\frac{dX_j^\varepsilon}{dt} = -\frac{Uh}{2\pi U} D^2 Y_j^\varepsilon$$

with FFT

$$\frac{dX_j^\varepsilon}{dt} = -\frac{Uh}{2\pi} \sum_{m=0}^{N-1} \frac{(Y_j^\varepsilon - Y_m^\varepsilon)}{\sin^2(\frac{h}{2}(j-m))}$$

$m+j \text{ odd}$

$$\frac{dY_j^\varepsilon}{dt} = -\frac{Uh}{2\pi} \sum_{\substack{m=0 \\ m+j \text{ odd}}}^{N-1} \left[\frac{X_j^\varepsilon - X_m^\varepsilon}{\sin^2(\frac{h}{2}(j-m))} - 2D\Gamma_j^\varepsilon \cot\left(\frac{h}{2}(j-m)\right) \right]$$

Baker & N. SISC '98

ATR = Alternate Trapezoid Rule,
very accurate for singular integrals

MORE on the HILBERT TRANSFORM

Take this complex potential
(A Cauchy-type integral)

$$\boxed{\Phi(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(x)}{x-z} dx},$$

$f(x) \rightarrow 0 \rightsquigarrow |x| \rightarrow \infty$, f smooth and

say $f(x) = \Theta(1/x^2)$

an upper analytic function (UHP)

a lower analytic function (LHP)

Let $z = 3+i\Im$, $\Im \geq 0 \Rightarrow \text{UHP} = \text{upper half-plane}$

$$\Phi(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(x)(x-\bar{z})}{|x-z|^2} dx = \dots \quad (\text{rationalize})$$

$$\dots = \frac{1}{\pi} \int_{\mathbb{R}} -\frac{f(x)\Im}{(x-3)^2 + \Im^2} dx - i \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)(x-\bar{3})}{(x-3)^2 + \Im^2} dx$$

$$= \boxed{\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x) \frac{dx/\Im}{1 + \left(\frac{x-3}{\Im}\right)^2}}{1 + \left(\frac{x-3}{\Im}\right)^2} dx} - i \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)}{(x-3) + \frac{\Im^2}{(x-3)}} dx$$

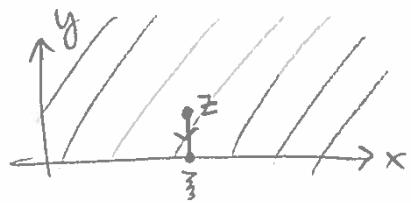
let $\frac{x-3}{\Im} = r$, $\frac{dx}{\Im} = dr$

Note that

$$\boxed{\frac{1}{\pi \Im} \int_{\mathbb{R}} \frac{1}{1 + \left(\frac{x-3}{\Im}\right)^2} dx} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1+r^2} dr = 1$$

will tend to a delta function as $3 \downarrow 0$.

Therefore we have that



$$\lim_{\zeta \downarrow 0} \Phi(\zeta) = f(\zeta) + i \mathcal{H}[f](\zeta) = f(z) + i g(z)$$

$\Phi(\zeta)$

↑ harmonic conjugate of $f(x, y)$

where
$$\mathcal{H}[f](\zeta) = \frac{1}{\pi} \int_R \frac{f(x)}{\zeta - x} dx .$$

- the Hilbert transform takes the boundary value of the harmonic func. $u(x_1, 0) = f(x)$ onto the boundary value of its harmonic conjugate $v(x_1, 0) = g(x)$.

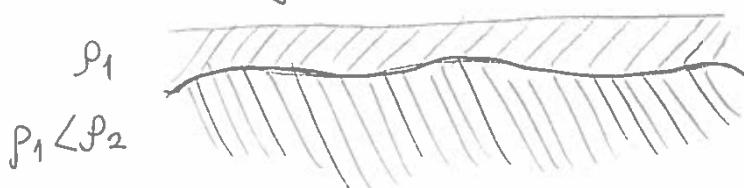
- from Cauchy-Riemann equations
- $$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$
- and the fact that $\frac{d}{dx} \mathcal{H}[f](x) = \mathcal{H}\left[\frac{df}{dx}\right](x)$

we can use the Hilbert transform as a Dirichlet-to-Neumann map on the half-plane, say an infinite depth domain.

- Hilbert transform is present in PDEs such as for INTERNAL WAVES

$$\eta_t + \eta \eta_x + \mathcal{H}[\eta_{xx}] = 0$$

Benjamin-Ono equation
for internal waves.



C (●) Comments on Homogenization Theory.

- Arises in problems in which we want to obtain a macroscopic or homogenized or effective equation for systems with a fine microscopic structure.
- Ideas started with second order elliptic problems as will be described. But is not restricted to this class of problems.

② Variable coefficient elliptic problem

$$-\nabla \cdot [A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon] = f(x), \quad \varepsilon \ll 1$$

A = rapidly varying conductivity (matrix)

u = density of a conserved quantity (balanced)

$$-A \nabla u = \text{flux}$$

Source term

Thermal conduct

Steady Heat Flux $q = -A \nabla T$ (Fourier law)

Porous media

$v = -A \nabla p$ (Darcy law)

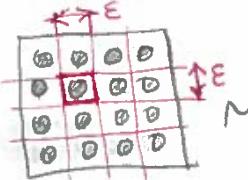
$f = 0$ incompress.

A depends on permeability

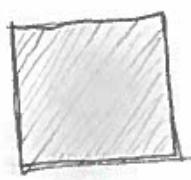
When $\varepsilon \rightarrow 0$, $u^\varepsilon \xrightarrow{\text{weakly}} u$ where the macroscopic (effective) solution satisfies:

$$-\nabla \cdot [A_H \nabla u] = f(x)$$

↑ effective conductivity



A
Composite Material



A_H
Effective material

Heterogeneous medium

Homogeneous medium

- Sketch of the math set-up...

③ Problem set in a (2D)

Multiple-Scale framework:

$$-\frac{\partial}{\partial x_1} \left[a_{11} u_{x_1} + a_{12} u_{x_2} \right] - \frac{\partial}{\partial x_2} \left[a_{21} u_{x_1} + a_{22} u_{x_2} \right] = f\left(\vec{x}, \frac{\vec{x}}{\varepsilon}\right)$$

where $u = u\left(\vec{x}, \frac{\vec{x}}{\varepsilon}\right) = u\left(\vec{x}, \vec{y}\right)$

$$a_{ij} = a_{ij}\left(\vec{x}, \frac{\vec{x}}{\varepsilon}\right)$$

think of y as an independent variable (due to scale separation)
but want to study how these MULTIPLE SCALES affect the problem

macroscale variation
microstructure.

Look for solution of the form
(different notation)

$$u^\varepsilon(\vec{x}) = u(\vec{x}; \varepsilon) = v(\vec{x}, \frac{\vec{x}}{\varepsilon}; \varepsilon) = v(\vec{x}, \vec{y}; \varepsilon)$$

$\uparrow \uparrow$
2 scales

where $u_{x_i} = \frac{\partial v}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial v}{\partial y_i}$

Plug this into equation:

$$\sum_i \left\{ -\frac{1}{\varepsilon^2} \frac{\partial}{\partial y_i} \left[\sum_j a_{ij} \frac{\partial v}{\partial y_j} \right] - \frac{1}{\varepsilon} \frac{\partial}{\partial x_i} \left[\sum_j a_{ij} \frac{\partial v}{\partial y_j} \right] - \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \left[a_{ij} \frac{\partial v}{\partial x_j} \right] - \frac{\partial}{\partial x_i} \left[a_{ij} \frac{\partial v}{\partial x_j} \right] \right\} = f(\vec{x}, \vec{y})$$

Now use a power series expansion

$$v(\vec{x}, \vec{y}; \varepsilon) = v_0(\vec{x}, \vec{y}) + \varepsilon v_1(\vec{x}, \vec{y}) + \varepsilon^2 v_2(\vec{x}, \vec{y}) + \dots$$

- One finds that the leading order term does depend on \vec{y} , does not feel the microstructure: $v_0 = v_0(\vec{x})$.
- to the next order one gets all problems for the auxiliary periodic functions w_k :

⊗
$$-\sum_i \frac{\partial}{\partial y_i} \left[\sum_j a_{ij} \frac{\partial w_k}{\partial y_j} \right] = \sum_i \frac{\partial a_{ik}}{\partial y_i}$$

Also that

$$a_{ij}^+ = \left\langle a_{ij} + \sum_k a_{ik} \frac{\partial w_k}{\partial y_k} \right\rangle$$

effective conductivity

there is like a
SEPARATION of
VARIABLES etc
 $K=1, 2$

- 1D simplification / example. Eq. ~~⊗~~ becomes

$$\boxed{-\frac{d}{dy} \left(a \frac{dw}{dy} \right) = \frac{da}{dy}}$$

Integrate

$\oplus \quad -a \frac{dw}{dy} + k = a \quad , \quad k = \text{constant of integration}$

$$\boxed{\frac{dw}{dy} = -1 + \frac{k}{a}}$$

Cell-average: recall w is cell-periodic

$$\langle \frac{dw}{dy} \rangle = \langle -1 \rangle + \langle \frac{k}{a} \rangle$$

$$0 = -1 + k \langle \frac{1}{a} \rangle$$

$$\Rightarrow \boxed{k = \frac{1}{\langle 1/a \rangle}} \quad \text{harmonic mean}$$

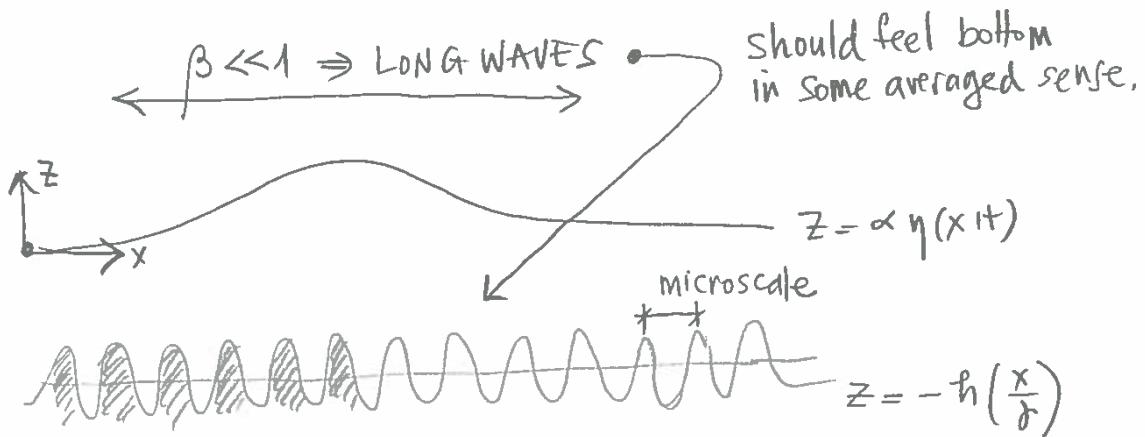
From above

$$q^H = \langle a + a \frac{\partial w}{\partial y} \rangle = k \quad \text{using } \oplus$$

$$\boxed{q^H = \frac{1}{\langle 1/a \rangle}}$$

not the arithmetic mean!

⑥ EFFECTIVE WAVE BEHAVIOR in a heterogeneous medium



$$\alpha = \frac{a}{h_0}, \quad \beta = \frac{h_0^2}{\lambda^2}, \quad \gamma = \frac{e}{\lambda}$$

$$\beta \phi_{xx} + \phi_{tz} = 0 \quad -h\left(\frac{x}{\lambda}\right) < z < \alpha \eta(x,t)$$

$$\begin{cases} \eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_z = 0 \\ \eta + \phi_t + \frac{\alpha}{2} (\phi_x^2 + \frac{1}{\beta} \phi_z^2) = 0 \end{cases} \quad t = \alpha \eta(x,t)$$

$$\phi_z + \frac{\beta}{\gamma} h'\left(\frac{x}{\lambda}\right) \phi_z = 0, \quad z = -h\left(\frac{x}{\lambda}\right),$$

Scale ordering is $\alpha = \Theta(\varepsilon)$, $\beta = \varepsilon^{(*)}$ and $\gamma = \sqrt{\varepsilon}$.

This is a SCALING that works leading to an
EFFECTIVE KdV-equation as we will see:
weakly nonlinear, weakly dispersive waves in the
presence of a rapidly varying periodic topography

Reference: Rozales and Papamichael, Studies in Applied Math.,
(MIT Journal), 1983.

(*) In the paper they use $\beta = \delta \varepsilon$, but I made $\delta = 1$.

C

lets use the multiple-scales power series expansion
(as our ansatz) :

D₂

$$\begin{aligned}\Phi(x, z, t) &= \phi_0(x, \bar{z}, z, \bar{t}) + \sqrt{\varepsilon} \phi_1(x, \bar{z}, z, \bar{t}) + \varepsilon \phi_2(x, \bar{z}, z, \bar{t}) + \dots \\ \eta(x, t) &= \eta_0(x, \bar{z}, \bar{t}) + \sqrt{\varepsilon} \eta_1(x, \bar{z}, \bar{t}) + \varepsilon \eta_2(x, \bar{z}, \bar{t}) + \dots\end{aligned}$$

$$\bar{t} = \varepsilon t, \quad x = x - C t \quad \text{e} \quad \bar{z} = x / \sqrt{\varepsilon}.$$

C

We are looking for a traveling-wave model in the presence of a spatial microstructure (in 3); in analogy with y of the composite material problem) having also a slow time (6); say for an effective behavior to appear on the "long run").

The ^{*}traveling wave speed C is unknown and will be calculated in the process, a bit like the effective conductivity. (\star effective)

We will take the depth variations to be such that

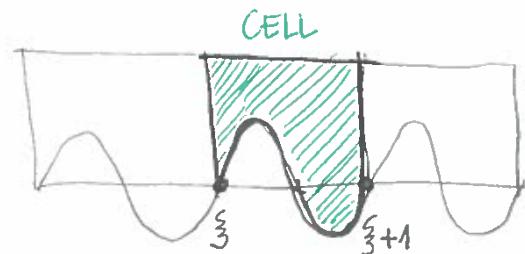
$$h(\bar{s}+1) = h(\bar{s})$$

C

namely 1-periodic in the fast variable \bar{s} .

So we have this "mathematical bi-focal lens" to see the microscale and the reference (wave) scale interact, if the scaling is well ordered.

Substitute the expansions \square_2 into our system \square_1 , which will yield a family of cell-problems ordered by powers of ε .



$$0 < \xi < 1, -h(\xi) < z < 0.$$

- $\mathcal{O}(\varepsilon^{1/2})$ -PROBLEM — (after collecting terms of the same order)

\square_3

$$\phi_{zzz} + \phi_{fzz} = \tilde{a}(\xi, z; x, \bar{b}), \quad \text{in the fluid}$$

$$\phi_{zz} + h_z \phi_{z\xi} = \tilde{b}(\xi; x, \bar{b}), \quad \text{at the bottom}$$

$$\phi_{zz} = \tilde{c}(\xi; x, \bar{b}), \quad \text{at the undisturbed F.S.}$$

- We use Separation of variables $\mathbf{F}(\xi, z) \Psi(x, \bar{b})$ to get CELL-PROBLEMS of the form!

"Roman letters" solve the CELL PROBLEM

\square_4

$$F_{zz} + F_{zzz} = a(\xi, z), \quad \text{in the CELL}$$

$$L[F] = F_z + h_z F_\xi = b(\xi), \quad \text{bottom of CELL}$$

$$F_z = c(\xi), \quad \text{top of CELL}$$

Right hand side will indicate that:

$$\phi_0 \equiv \tilde{\Psi}_0^*(x, \bar{b})$$

$$\phi_1 \equiv \tilde{\Psi}_1^*(x, \bar{b}) + \tilde{\mathbf{A}}(\xi, z) \tilde{\Psi}_1(x, \bar{b})$$

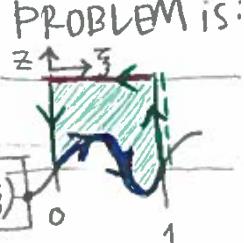
etc...

— All the forcing terms in \square_4 (RTTs) are \tilde{z} -periodic.

— We have a family of elliptic Poisson-type cell problems, at each $O(\varepsilon^{d/2})$ -level.

— The compatibility (solvability) condition for the CELL PROBLEM is:

$$\square_5 \quad \langle a \rangle = \int_0^1 \int_{-h(\tilde{z})}^0 a(\tilde{z}, z) dz d\tilde{z} = \int_0^1 [c(\tilde{z}) - b(\tilde{z})] d\tilde{z}.$$



We can call this a cell average even though it is not normalized by the cell area.

$$\frac{\langle a \rangle}{\langle 1 \rangle}$$

\square_5'

— The compatibility condition of the $O(\varepsilon)$ cell-problem leads to the expression for the EFFECTIVE PROPAGATION SPEED:

$$\square_6 \quad G^2 = 1 - \langle A_{\tilde{z}}^2(\tilde{z}, z) + A_z^2(\tilde{z}, z) \rangle \quad \square_1$$

Remarks:

1 Speed value depends on the cell-average of the auxiliary function $A(\tilde{z}, z)$.

2 Even if the periodic bottom is MEAN-ZERO the WAVE SLOWS DOWN! "smaller conductivity" (in analogy w/ before)

The following solution-structure arises

$\tilde{z}=0$ @ $O(1)$ —

$$\phi_0 \equiv \psi_0(x, \tilde{z})$$

$\tilde{z}=1$ @ $O(\sqrt{\varepsilon})$ —

$$\phi_1 \equiv \psi_1(x, \tilde{z}) + A(\tilde{z}, z) \psi_0(x, \tilde{z})$$

$\tilde{z}=2$ @ $O(\varepsilon)$ —

$$\phi_2 \equiv \psi_2(x, \tilde{z}) + A(\tilde{z}, z) \psi_1(x, \tilde{z}) + B(\tilde{z}, z) \psi_0(x, \tilde{z})$$

Use linearity of \square_3 etc...

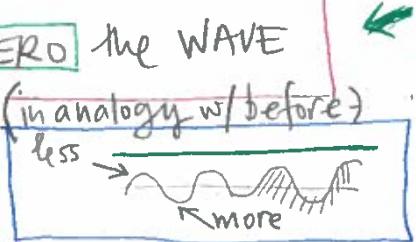
Say

$$\begin{aligned} \Delta \phi_1 &= 0 \\ L[\phi_1] &= \tilde{b}_1 + \tilde{b}_2 \\ \phi_1|_{z=0} &= 0 \end{aligned}$$

to get

$$\begin{aligned} \Delta A &= 0 \\ L[A] &= -\tilde{h}_3 \\ A|_{z=0} &= 0 \end{aligned}$$

$$\begin{cases} \tilde{b}_1 = 0 \\ \tilde{b}_2 = -\tilde{h}_3 \psi_0(x) \end{cases}$$



— THREE REGIMES —

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"regime A"
(micro) - In the limit as $\epsilon \rightarrow 0$ the channel will be

analogous to a well-mixed composite material with waves moving to the right with an effective speed equal to G (\square_6) and therefore no reflexion being generated.

"regime B"
(intermediate) It is worth mentioning that the strongest wave reflexion regime is when $2\lambda_b = \lambda$, where λ is the linear wavelength and λ_b is the period of the bottom variations. This is the regime for BRAGG RESONANCE.

"regime C"
(macro) - And there is also the regime of slowly varying topography where one gets phase variations as seen in geometrical optics for waves.



\square_4 - Finally the compatibility (solvability) condition in the $\Theta(\epsilon^2)$ problem leads to an effective equation for $\eta_0(x, z)$. Noticing (in the process) that

$$\eta_0 = G \eta_0(x, z) / z$$

- one arrives at the EFFECTIVE kdv equation for the wave elevation at leading order:

\square_7

$$\eta_{0z} + \Theta(\eta_0^2)_{zz} + b \eta_{0xxx} = 0$$

where

$$\Theta = G^{-1} \left\{ \frac{3}{4} \alpha_0 + \frac{1}{4} \alpha_0 \langle A_s^2 \rangle_s - \frac{1}{2G} \langle h_3 D \rangle_b \right\}$$

$$b = G^{-1} \left\{ -\frac{1}{2} \langle h_3 C \rangle_b - \frac{1}{2} G^2 \langle B_s \rangle_s \right\}$$

bottom

surface

effective (inertial)
nonlinear coeff

effective dispersion c

R&P'83: When $h(z) = 1$ (flat) \square_7 reduces to usual kdv.

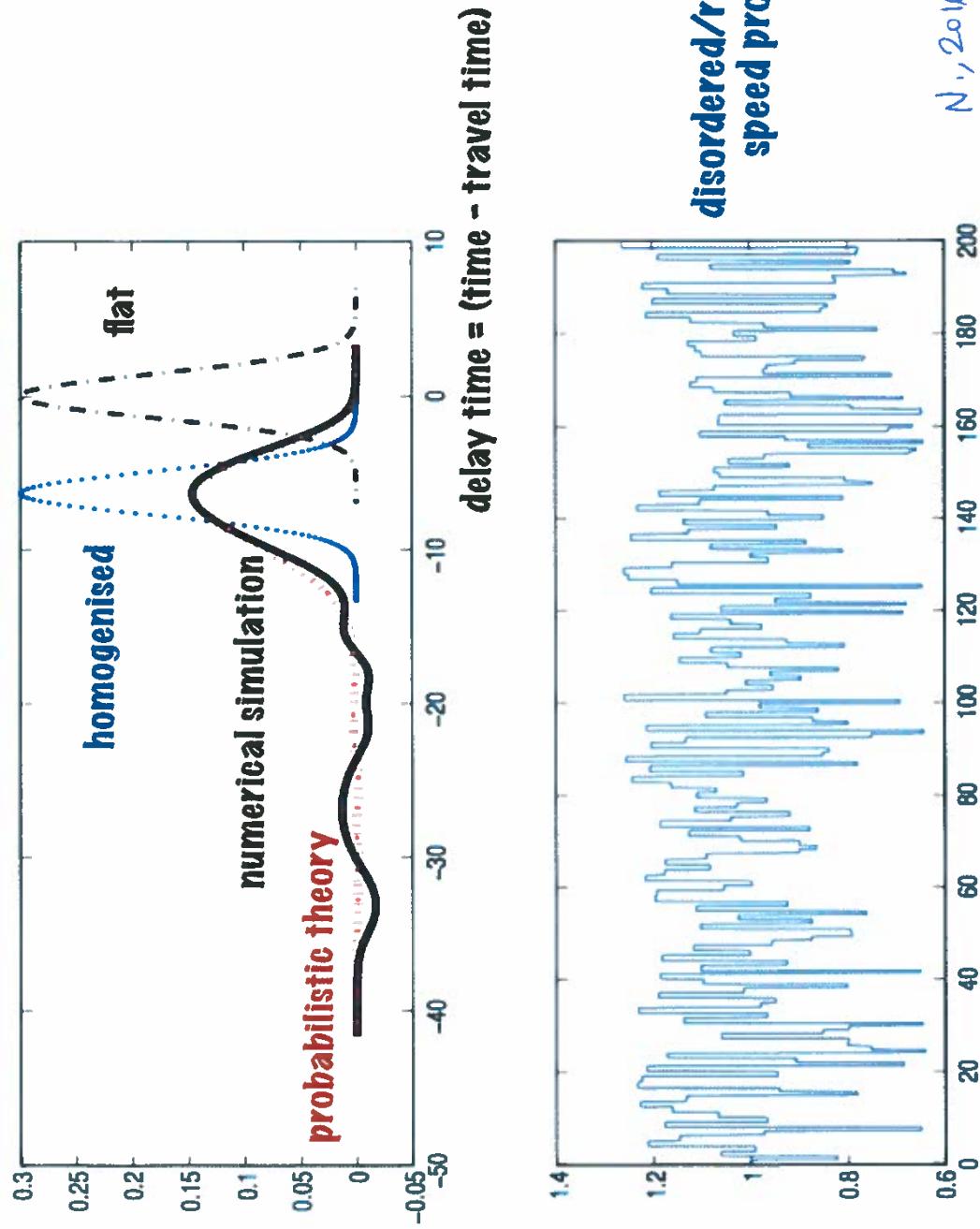


FIG. 1.2. Bottom: a realisation for a piecewise-constant random propagation speed centred about the background unit-speed. There are 200 layers. Top: The horizontal axis is in terms of delay-time. Zero delay-time means the wave arrived on time regarding the underlying homogeneous system. The dash-dotted profile depicts the travelling wave solution of the underlying homogeneous hyperbolic system. The solution of the homogenised system is given by the dotted pulse profile. The solid line is for the solution of the hyperbolic system, with the small scale random fluctuations

The NLS - nonlinear Schrödinger eqn.

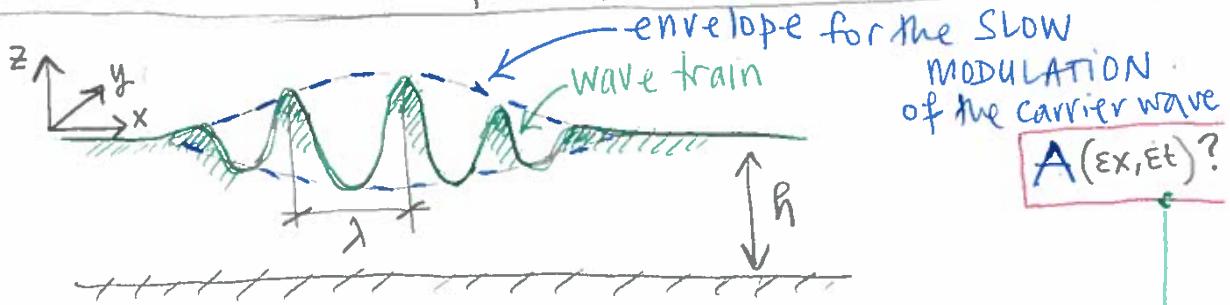
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Evolution Equations for SLOWLY MODULATED weakly nonlinear waves - highlights of a lengthy calculation

• Zakharov 1968, Benney & Roskes 1969

Reference: C.C. Mei, M. Sjorsse and D. Yue
(book)

Theory and Applications of Ocean Surface Waves,
World Scientific, 2005, PAGE 3747.



Potential Theory - MAIN IDEAS - MAIN RESULTS.

• MULTIPLE SCALE analysis.

regime: intermediate depth ; not long wave regime
(h/λ) - not very small

The SMALL PARAMETER

slow modulation:

- AMPLITUDE MODULATION ANALYSIS:

$$KA = \epsilon \text{ small}$$

$$KA = 2\pi \frac{A}{\lambda} = O(\epsilon)$$

Set $\begin{cases} x, x_1 = \epsilon x, x_2 = \epsilon^2 x, \dots \\ y, y_1 = \epsilon y, y_2 = \epsilon^2 y, \dots \\ t, t_1 = \epsilon t, t_2 = \epsilon^2 t, \dots \end{cases}$

dominant propagation direction
weak lateral dependence
(may not need all these SCALES)

Use
$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi_n, \eta = \sum_{n=1}^{\infty} \epsilon^n \eta_n$$

where
$$\begin{cases} \phi_n = \phi_n(x, x_1, x_2, \dots; y, y_1, \dots; t, t_1, t_2, \dots), \\ \eta_n = \eta_n(x, x_1, x_2, \dots; y, y_1, \dots; t, t_1, t_2, \dots) \end{cases}$$

- To get the following family of linear PDE system:

①

 $\Theta(\varepsilon^n)$ -equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \phi_n = F_n, \quad -h < z < 0$$

☒
1

$$\left(\frac{\partial^2}{\partial t^2} + g \frac{\partial}{\partial z} \right) \phi_n = G_n, \quad z = 0$$

$$\frac{\partial \phi_n}{\partial z} = 0, \quad z = -h$$

$$\eta_n = -\frac{1}{g} H_n, \quad z = 0$$

where F_n, G_n and H_n depend on $\phi_m, m < n$, and its derivatives. (lengthy expressions)

- Since the carrier wave is monochromatic (single spatial frequency) let's consider the ansatz

$$\boxed{2} \quad \phi_n = \sum_{m=-n}^n e^{im\psi} \phi_{nm}$$

where $\psi = \cancel{k}x - wt$ is the phase related to the underlying carrier wave.

Remarks :

(“slowly varying Fourier modes” amplitudes)

- ① $\phi_{nm} = \phi_{nm}(x_1, x_2, \dots; y_1, y_2, \dots; z; t_1, t_2, \dots)$
- ② F_n, G_n inherits the same structure as ~~ϕ~~ ₂ above
- ③ $\phi_{n,-m} = \overline{\phi_{nm}}$ so that ϕ is real valued

- ④ Why $\phi_n = \sum_{m=-n}^n$? Due to nonlinearity and Fourier mode degeneration for higher order terms.
(quadratic terms in G_n)

□ The family of problems become as follows.
Each $O(\varepsilon^n)$ -problem ~~ϕ~~ ₁ will lead to the following
ODEs - along the vertical coordinate :

①

(nm)- BOUNDARY VALUE PROBLEMSturm-Liouville
type problems

$$\left(\frac{\partial^2}{\partial z^2} - m^2 k^2 \right) \phi_{nm} = F_{nm}, \quad -h < z < 0$$

$$\left(g \frac{\partial}{\partial z} - m^2 \omega^2 \right) \phi_{nm} = G_{nm}, \quad z = 0$$

$$\frac{\partial}{\partial z} \phi_{nm} = 0, \quad z = -h$$

The “m-steps” are solved sequentially to obtain ~~ϕ~~ ₂ for a given n .

Remark :

$$\omega^2 = k g \tanh kh$$

 k = related to carrier wave.

First steps:

$$F_{10} = G_{10} = F_{11} = G_{11} = 0$$

$M=0$

$$\frac{\partial^2}{\partial z^2} \phi_{10} = 0, \quad -h \leq z \leq 0$$

$$g \frac{\partial}{\partial z} \phi_{10} = 0, \quad z = 0$$

$$\frac{\partial}{\partial z} \phi_{10} = 0, \quad z = -h$$

We conclude that ϕ_{10} doesn't depend on z , and

$$\phi_{10} = \phi_{10}(x_1, x_2, \dots, y_1, y_2, \dots, t_1, t_2, \dots) = \bar{\phi}_{10} \quad (M=0).$$

Also

$$\phi_1 = \phi_{10} - \frac{g \cosh(k(z+h))}{2w \cosh(kh)} \left(i A e^{iz\omega} + * \right)$$

Where $*$ = complex conjugate

$$A = A(x_1, x_2, y_1, y_2, t_1, t_2, \dots)$$

We will get an equation for A .

Slowly varying amplitude of this mode.

Solvability conditions:

Take $m=1$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \phi_{m1} = F_m, \quad -h \leq z \leq 0$$

$$\left(g \frac{\partial}{\partial z} - \omega^2 \right) \phi_{m1} = G_m, \quad z = 0$$

$$\frac{\partial}{\partial z} \phi_{m1} = 0, \quad z = -h$$

① First look at the homogeneous solution:

$$\begin{aligned} \left(\frac{\partial^2}{\partial z^2} - k^2 \right) f &= 0 & -h < z < 0 \\ \left(g \frac{\partial}{\partial z} - \omega^2 \right) f &= 0 & z = 0 \\ \frac{\partial f}{\partial z} &= 0 & z = -h \end{aligned}$$

It is easy to see that $f(z) = \frac{\cosh(k(z+h))}{\cosh kh}$ is a solution:

② The solvability (compatibility) condition for $\underset{n,m=1}{\exists}$ is obtained from integrating by parts:

$$\begin{aligned} \int_{-h}^0 \left\{ f \left[\frac{\partial^2}{\partial z^2} \phi_m - k^2 \phi_{m1} \right] - \phi_{m1} \left[\frac{\partial^2}{\partial z^2} f - k^2 f \right] \right\} dz &= \\ &= \left[f \frac{\partial \phi_{m1}}{\partial z} - \phi_{m1} \frac{\partial f}{\partial z} \right]_{-h}^0 \end{aligned}$$

Therefore

$$\int_{-h}^0 f F_{m1} dz = \left[f \left(\frac{1}{g} G_{m1} + \omega^2 \phi_{m1} \right) - \phi_{m1} \left(\frac{1}{g} \cdot 0 + \omega^2 f \right) \right]_0^0$$

$$\int_{-h}^0 \frac{\cosh(k(z+h))}{\cosh kh} F_{m1} dz = \frac{1}{g} G_{m1}, \quad f(0) = 1 \quad \text{--- C}$$

- From the compatibility cond. with $n=2, m=1$ we obtain

C

$$\frac{\partial A}{\partial t_1} + C_g \frac{\partial A}{\partial x_1} = 0$$

Remark: It can be shown (Whitham '74) that energy also propagates with GROUP SPEED

where the group speed is $C_g = \frac{\partial \omega}{\partial k}$

- To leading order the amplitude (of envelope) travels with group speed while the carrier wave propagates with phase speed $C(k) = \frac{\omega}{k}$ as in 7F.

- From the solvability condition for $n=3, m=1$, performing some manipulations and imposing initial and far field conditions, one obtains that

$$i \frac{\partial A}{\partial \xi} + \frac{\omega''}{2} \frac{\partial^2 A}{\partial \xi^2} + \beta |A|^2 A = 0$$

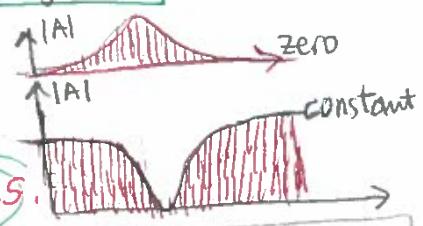
namely the cubic nonlinear Schrödinger equation. (NLS)

Here $\xi = x_1 - C_g t_1$, $\zeta = \epsilon t_1$, $\beta = \beta(k)$

* $\begin{cases} \beta > 0, \text{ focusing} \Rightarrow \text{"bright" solitons} \\ \beta < 0, \text{ defocusing} \Rightarrow \text{"dark" solitons} \end{cases}$

(Ablowitz pag 132, Nonlin. Disp. Wave 2011)

terminology from OPTICS.



$$\beta(k) = -\frac{\omega k^2}{16 \sinh^4 kh} (\cosh 4kh + 8 - 2 \tanh^2 kh) + \frac{\omega}{2 \sinh^2 2kh} \frac{(2\omega \cosh^2 kh + k C_g^2)}{(gh - C_g^2)}$$

$\beta(k)$ \leftarrow phase.

A in complex = $a e^{i\phi}$
amplitude

OPTICS: Solitons are in

$$a = |A|$$

optical bit stream



Heat equation - Diffusion equation

$$u_t = \kappa \Delta u, \quad \kappa - \text{diffusion coeff.}, \quad \kappa > 0.$$

$$u = u(x_1, x_2, \dots, x_n, t)$$

Take the (normalized) dimensionless eq. $u_t = \Delta u$ and a Fourier Mode: $u = e^{i(x \cdot \xi + \omega t)}$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) = \text{wavenumber vector}$

Clearly $i\omega = -|\xi|^2$ is the dispersion relation; so that

$$u = e^{-|\xi|^2 t} e^{i(x \cdot \xi)}$$

amplitude decays exponentially, and the RATE depends on the WAVENUMBER.

Question: what happens with the BACKWARD heat equation?

Take the normalized heat equation (where $\kappa = 1$).

$$\begin{aligned} u_t - \Delta u &= 0, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u &= f(x), \quad t = 0, \quad x \in \mathbb{R}^n \end{aligned}$$

let $f(x)$ be continuous. With a Fourier representation do we have a solution $u \in C^2(\mathbb{R}^n, t > 0)$ where $u \in C(\mathbb{R}^n, t \geq 0)$?

Based on the dispersion relation we write

1 $u(x, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{f}(\xi) d\xi$

where the Fourier transform is

2 $\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy \cdot \xi} f(y) dy$

and the inverse transform is

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) d\xi$$

using \mathcal{D}_2 in expression \mathcal{D}_1 and admitting we all the conditions to manipulate the integrals accordingly, we have that

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t) f(y) dy$$

$$K(x, y, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi - |\xi|^2 t} d\xi$$

This integral can be evaluated by completing the "square":

$$i(x-y) \cdot \xi - |\xi|^2 t = -\left(-i \frac{(x-y)}{2\sqrt{t}} + \frac{3}{2}\sqrt{t}, -i \frac{(x-y)}{2\sqrt{t}} + \frac{3}{2}\sqrt{t}\right) +$$

$$+ \left(-i \frac{(x-y)}{2\sqrt{t}}, -i \frac{(x-y)}{2\sqrt{t}}\right) =$$

$$= -(\eta_0, \eta_0) - \frac{1}{4t} ((x-y), (x-y))$$

where

$$\eta_0 = -i \frac{(x-y)}{2\sqrt{t}} + \frac{3}{2}\sqrt{t}$$

Therefore

$$\begin{aligned} K(x, y, t) &= (2\pi)^{-n} \int e^{-|\eta|^2} e^{-|x-y|^2/4t} t^{-n/2} d\eta = \\ &= (4\pi^2 t)^{-n/2} e^{-|x-y|^2/4t} \boxed{\int e^{-|\eta|^2} d\eta} \\ &\quad \underbrace{(\pi)^{n/2}}_{(4\pi^2 t)^{-n/2}}. \end{aligned}$$

and

$$K(x, y, t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

is the GAUSSIAN KERNEL.

Note: $\int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta = \left(\int_{-\infty}^{\infty} e^{-s^2} ds \right)^n = (\pi)^{n/2}$

- ① Let $f(x)$ be continuous and bounded in \mathbb{R}^n .
Then

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t) f(y) dy = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy$$

is $C^\infty(\mathbb{R}^n, t > 0)$ & satisfies $u_t = \Delta u, t > 0$.

Also when $\lim_{t \downarrow 0} u(x, t) = f(x)$

- ⑥ The procedure to check these properties is not that different from what we did with Laplace's equation.
We use the following properties:

- (a) $K(x, y|t) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n, t > 0)$ smooth heat kernel.
- (b) $\left(\frac{\partial}{\partial t} - \Delta \right) K(x, y|t) = 0, t > 0.$
- (c) $K(x, y|t) > 0, t > 0.$
- (d) $\int K(x, y|t) dy = 1, x \in \mathbb{R}^n, t > 0$ total mass one
- (e)

$$\boxed{\lim_{t \rightarrow 0^+} \int_{|y-x|>\delta} K(x, y|t) dy = 0}$$

uniformly in $x \in \mathbb{R}^n$

(d) + (e) combine to give a **Dirac delta-like effect**
as $t \downarrow 0$.

Details are given in page 146A.

Checking (e): Consider the integral away from the singularity. Then

$$\begin{aligned} \int_{|y-x|>\delta} K(x, y, t) dy &= (4\pi t)^{-n/2} \int_{|y-x|>\delta} e^{-|x-y|^2/4t} dy = \\ &= (4\pi t)^{-n/2} \int_{|\eta|>\frac{\delta}{(4t)^{1/2}}} e^{-|\eta|^2/(4t)} d\eta = \pi^{-n/2} \int_{|\eta|>\frac{\delta}{(4t)^{1/2}}} e^{-|\eta|^2} d\eta \end{aligned}$$

Clearly for $\delta > 0$ and $t \rightarrow 0$ the integral "catches" only the exponentially small tails of the Gaussian.

— Checking the initial condition

$$\begin{aligned} |u(x, t) - f(\xi)| &= \left| \int K(x, y, t) (f(y) - f(\xi)) dy \right| \leq \\ &\leq \int_{|y-x|<\delta} + \int_{|y-x|>\delta} K(x, y, t) |f(y) - f(\xi)| dy \leq \\ &\leq \underbrace{\varepsilon \int_{|y-x|<\delta} K(x, y, t) dy}_{< 1, \text{ by (c)+(d)}} + 2M \underbrace{\int_{|y-x|>\delta} K(x, y, t) dy}_{< \frac{\varepsilon}{2M}, \text{ by (e)}} \leq 2\varepsilon \end{aligned}$$

Burgers equation = advection-diffusion

COLE-HOPF transformation ('50s)

Burgers eqn.

(Kerkin pg 31, Whitham Cap 4)

$$\#_A \quad u_t + u u_x - \varepsilon u_{xx} = 0, \quad \varepsilon > 0, \quad \text{advection-diffusion eq}$$

Cole-Hopf change of variables :

$$u \equiv -2\varepsilon \frac{v_x}{v}$$

(Cole-Hopf)

$$u_t = -2\varepsilon \frac{v_{xt}}{v} + 2\varepsilon \frac{v_x v_t}{v^2}$$

$$u_{xx} = -2\varepsilon \frac{v_{xx}}{v} + 2\varepsilon \frac{v_{x^2}}{v^2}$$

$$u_{xxy} = -2\varepsilon \frac{v_{xxx}}{v} + 6\varepsilon \frac{v_x v_{xx}}{v^2} - 4\varepsilon \frac{v_x^3}{v^3}$$

Substituting in Burgers eq. yields

$$\#_B \quad \frac{\sqrt{x}}{v} (E v_{xx} - v_t) - (E v_{xx} - v_t)_x = 0$$

= 0 by another factor

therefore when v satisfies the LINEAR heat equation

$$\boxed{v_t = E v_{xx}}$$

(in $\#_B$), eq. $\#_A$ is automatically satisfied.

- Take the initial value problem

$$\begin{cases} u_t + uu_x - \epsilon u_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$

- Due to the Cole-Hopf transformation the initial condition becomes

$$f(x) = -\frac{2\epsilon v_x(x_0)}{v(x_0)}$$

- to get the corresponding initial condition solve the ODE:

$$v_x + \frac{f(x)}{2\epsilon} v = 0$$

- through the integrating factor we have

$$v(x,0) = g(x) = c \exp\left(-\frac{1}{2\epsilon} \int_0^x f(s) ds\right)$$

- Hence

$\textcircled{6}$
c = constant (will not matter)

$$\begin{aligned} v_t &= \epsilon v_{xx}, \quad -\infty < x < \infty \\ v(x,0) &= c g(x) \end{aligned}$$

which has as solution (page 145)

$$v(x,t) = \frac{c}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} g(y) e^{-(x-y)^2/4\epsilon t} dy$$

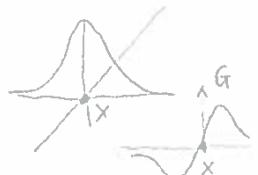
Recall $u = -2\epsilon \frac{v_x}{v}$

$$v(x,t) = \frac{-c}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} \left(\frac{g(y)(x-y)}{2\epsilon t} \right) e^{-(x-y)^2/4\epsilon t} dy$$

— We are immediately back to our original variable.

$$u(x,t) = \frac{\int_{-\infty}^{\infty} g(y) \frac{(x-y)}{t} e^{-(x-y)^2/4\epsilon t} dy}{\int_{-\infty}^{\infty} g(y) e^{-(x-y)^2/4\epsilon t} dy}$$

the constant c
does not matter,
it's gone...



⑥ using the expression for $g(x)$ (page 148) we can write

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-y}{t}\right) e^{-\frac{G(x,y,t)}{2\epsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{G(x,y,t)}{2\epsilon}} dy},$$

⑦ where $G(x,y,t) = \int_0^y f(s) ds + \frac{(x-y)^2}{4t}$

— Question of interest: What happens when $\epsilon \downarrow 0$?
What is the limiting form of the diffusive regularization?

— As we will see the regularized solution (prior to the shock)
converges to the inviscid ($\epsilon=0$) solution in a classical (strong)
sense. We recover the solution of the method of characteristics.

— First a few facts about integrals as in ⑦ above.

Asymptotics with integrals \Rightarrow their leading order contributions.

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- ① Laplace's Method for integrals with a parameter "m":

$$\text{#}_1 \quad I(m) = \int_a^b f(t) e^{-m\phi(t)} dt$$

Ablowitz & Fokas
page 422.

$\phi(t), f(t)$ smooth in $[a, b]$,

$\phi'(c) = 0, \phi''(c) > 0, a < c < b$.

- Intuition for main asymptotic contribution, which comes from the neighborhood of c .

Take

$$A = \int_{c-\varepsilon}^{c+\varepsilon} f(c) \exp \left\{ -m \left[\phi(c) + \frac{(t-c)^2}{2} \phi''(c) \right] \right\} dt$$

first Taylor terms

let $\bar{\tau} = \sqrt{\frac{m}{2}} \phi''(c) |(t-c)|$ so that

$$A = \frac{e^{-m\phi(c)}}{\sqrt{\frac{m}{2} \phi''(c)}} \int_{-\bar{\tau}_1}^{\bar{\tau}_1} e^{-\bar{\tau}^2} d\bar{\tau}$$

$$, \quad \bar{\tau}_1 = \varepsilon \sqrt{\frac{m}{2} \phi''(c)}$$

Recall that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, so for $m \gg 1$ regime of interest

$$\text{#}_2 \quad I(k) \sim f(c) \sqrt{\frac{2\pi}{m \phi''(c)}} e^{-m\phi(c)}$$

- ① Take a Fourier-type integral:

$$I(K) = \int_a^b f(t) e^{ikt} dt$$

$f(t)$ is smooth in $[a, b]$.

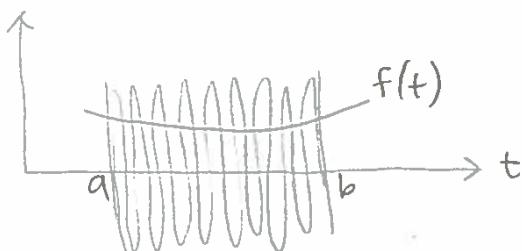
- If we take $f(t)$ to be a constant, say 1, then

$$I(K) = \int_a^b e^{ikt} dt = \left[\frac{e^{ikt}}{ik} \right]_a^b = -\frac{i}{K} [e^{ika} - e^{ikb}]$$

$$\lim_{K \rightarrow \infty} I(K) = 0$$

- Along these lines there is the Riemann-Lebesgue lemma:

$$\lim_{K \rightarrow \infty} \int_a^b f(t) e^{ikt} dt = 0, \text{ for } \int_a^b |f(t)| dt < \infty$$



roughly f is so slow with respect to oscillations that it looks almost (locally) like a constant.

- Fourier integrals of the form

$$I(K) = \int_a^b f(t) e^{ik\phi(t)} dt$$

leads to the method of Stationary Phase:

main contribution when

$$\phi'(t) = 0$$

More details and cases in references

Back to COLE-HOPF

- In the Burgers problem (page 149) we have that the stationary point satisfies

$$\frac{\partial G}{\partial y} = f(y) - \frac{(x-y)}{t} = 0.$$

Therefore the solution is $y = \bar{s}(x, t)$ so that

$$f(\bar{s}) - \frac{(x-\bar{s})}{t} = 0$$

Now studying the regime of vanishing viscosity

$\frac{1}{\varepsilon} \rightarrow \infty$ ($\varepsilon \rightarrow 0$), analogous to $m \rightarrow \infty$ (Laplace's method)

From
#1
#2
(page 150)

$$\int_{-\infty}^{\infty} g(y) e^{-\frac{G(y)}{2\varepsilon}} dy \sim g(\bar{s}) \sqrt{\frac{4\pi\varepsilon}{G''(\bar{s})}} e^{-\frac{G(\bar{s})}{2\varepsilon}}.$$

Applying to \bullet (page 149):

$$\int_{-\infty}^{\infty} \left(\frac{x-y}{t} \right) e^{-\frac{G(x,y,t)}{2\varepsilon}} dy \sim \left[\frac{(x-\bar{s})}{t} \right] \sqrt{\frac{4\pi\varepsilon}{G_{yy}(\bar{s}, t)}} e^{-\frac{G(x, \bar{s}, t)}{2\varepsilon}}$$

$$\int_{-\infty}^{\infty} e^{-\frac{G(x,y,t)}{2\varepsilon}} dy \sim \sqrt{\frac{4\pi\varepsilon}{G_{yy}(\bar{s}, t)}} e^{-\frac{G(x, \bar{s}, t)}{2\varepsilon}}$$

which simplifies to

as in method of charact

$$u(x, t) \sim \frac{x - \bar{s}(x, t)}{t} = f(\bar{s}(x, t)), \quad x = \bar{s} + \frac{f(\bar{s})}{t}, \quad \bar{s} = x - f(\bar{s})t$$

Remarks:

- we have recovered the (inviscid) Burgers solution (prior to a shock)
- the stationary point $\bar{s}(x, t)$ plays the role of the characteristic variable
- when a shock forms there are 2 stationary points and asymptotic analysis requires adjustments (Whitham, page 99)

① REMARKS on BOUNDARY LAYERS / singular perturbations

Ex1

Note that $y(x) = a(x-1) + 1$ is the solution to the first order problem. Note also that the $\epsilon=0$ does NOT have a solution

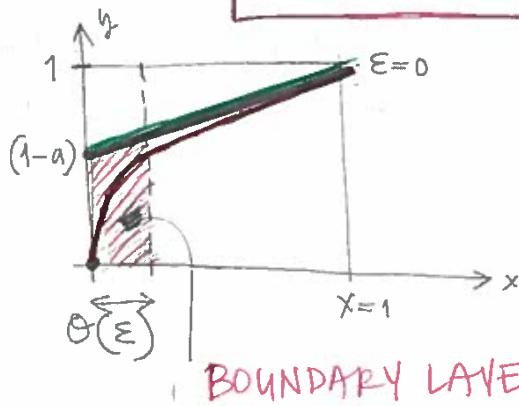
$$\left. \begin{array}{l} \epsilon \neq 0 \\ \epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = a \\ y(0) = 0 \\ y(1) = 1 \end{array} \right\}$$

First order ODE

$$\frac{dy}{dx} = a$$

$$y(1) = 1$$

$$y^\epsilon(x) = \frac{(1-a)(1-e^{-x/\epsilon})}{(1-e^{-1/\epsilon})} + ax$$



$$\text{OBS: } \epsilon=0 \Rightarrow y^\epsilon(x) = (1-a) + ax$$

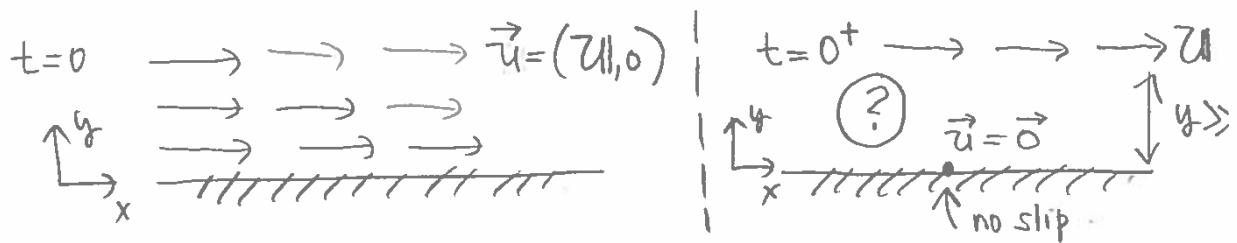
$$y_0^\epsilon(x) \leftarrow \lim_{\epsilon \rightarrow 0} y^\epsilon(x)$$

Note $\lim_{\epsilon \downarrow 0} y^\epsilon(0) \neq y(0)$.

Good reference in ODEs: Bender and Orszag, Advanced Mathematical Methods for Scientists and Engineers, 1978.

Ex 2: Consider the idealized flow problem in 2D.

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- Therefore we have the following boundary conditions:

$$\begin{cases} \vec{u} = \vec{0} \text{ at } y=0 \\ \lim_{y \rightarrow \infty} \vec{u} = (\vec{U}_0, 0) \end{cases}$$

- Take the dimensionless Navier-Stokes eq.:

$$(2D) \quad \boxed{u_t + (\vec{u} \cdot \nabla u)} = - \frac{\nabla p}{\rho} + \frac{1}{Re} \Delta u, \quad \begin{array}{l} v = \text{viscosity} \\ Re = \text{Reynolds number} \\ Re = \frac{U L}{v} \end{array}$$

- Use geometrical features to simplify the equations:

$$\boxed{\vec{u}(x, y, t) = (u(y, t), 0)} \quad \begin{array}{l} \text{no vertical component needed} \\ \text{no } x\text{-dependence (translation invariant in } x) \end{array}$$

- From \otimes_1 and \otimes_2 we have that

$$\boxed{u u_x + v u_y \equiv 0}, \quad \boxed{\frac{\partial p}{\partial x} = 0}$$

and the N-S eq. is reduced to this SCALAR diffusion eq.:

$$\boxed{u_t = \epsilon u_{yy}}, \quad \epsilon = \frac{1}{Re}$$

Let's see an alternative way to solve this DIFFUSION EQ. Even though we have dimensionless variables we can check the interplay of scales (space and time) or equivalently how to normalize the diffusion coeff.

(as mentioned in page 14)

$$\tilde{y} = \frac{y}{L}, \quad \tilde{t} = \frac{t}{T}$$

Scale in vertical direction

then

$$u_{\tilde{t}} = \frac{\varepsilon T}{L^2} u_{\tilde{y}\tilde{y}}$$

is the re-scaled diffusion equation which will indicate the length-scale associated with the BOUNDARY LAYER.

To normalize the diffusion coefficient

$$L = \sqrt{\varepsilon T}$$

and this leads to an ansatz for a SELF-SIMILAR change of variables

Let

$$\eta_b = \frac{y}{2\sqrt{\varepsilon t}} \quad \square_1$$

just for convenience.

Let's go back to our diffusion equation

$$u_t = \varepsilon u_{yy}, \quad \text{with} \quad f(\eta_b) = \frac{u(y, t)}{u_1} \quad \square_2$$

which yields an ODE:

$$f' \eta_t = \varepsilon f'' \eta_y^2$$

Which becomes (using the definition of η_b) \square_1

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$$-\frac{f'}{2t} \eta_b = f'' \frac{1}{4t} \Rightarrow \boxed{f'' + 2\eta_b f' = 0} \quad \begin{cases} f(0) = 0 \\ f(\infty) = 1 \end{cases}$$

using \square_2

the integration factor is $e^{\eta_b^2}$ and

$$f'(\eta) = c_1 e^{-\eta^2}$$

which is integrated to give

$$\boxed{f(\eta) = c_1 \int_0^{\eta_b} e^{-s^2} ds + c_2}$$

Using the boundary conditions gives

$$f(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-s^2} ds = \text{erf}(\eta_b),$$

the error function

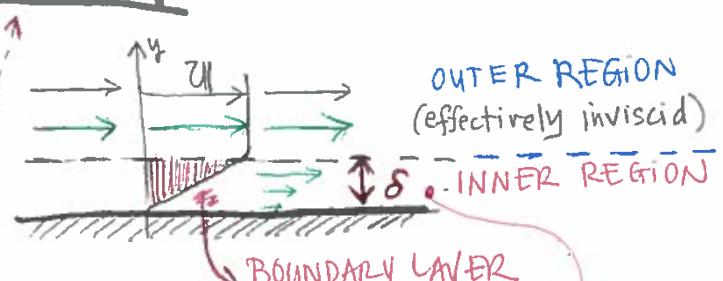
$$\begin{cases} \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \\ \text{erf}(\infty) = 1 \end{cases}$$

and substituting back to our original variables:

$$\boxed{u(y, t) = U \text{erf}\left(\frac{y \sqrt{R}}{2 \sqrt{t}}\right)} \Rightarrow \begin{cases} \text{double-checking} \\ u(0, t > 0) = 0 \\ u(y, t) \rightarrow U, y \rightarrow \infty \end{cases}$$

Thickness of BL
changes in time

$$\delta = O\left(\sqrt{\frac{t}{R}}\right)$$



Vorticity: $\omega = -uy$ ($y|_{x=0}$)

$$\omega = -\frac{2U}{\sqrt{\pi} 4\sqrt{t}} e^{-\frac{y^2}{4\sqrt{t}}}, \quad t > 0$$

- ① production of vorticity at the boundary.
- ② at $t=0^+$, boundary is like a vortex sheet: a Dirac delta distribution of vorticity.



13. Biological Waves: Single-Species Models

13.1 Background and the Travelling Waveform

There is a vast number of phenomena in biology in which a key element or precursor to a developmental process seems to be the appearance of a travelling wave of chemical concentration, mechanical deformation, electrical signal and so on. Looking at almost any film of a developing embryo it is hard not to be struck by the number of wavelike events that appear after fertilisation. Mechanical waves are perhaps the most obvious. There are, for example, both chemical and mechanical waves which propagate on the surface of many vertebrate eggs. In the case of the egg of the fish *Medaka* a calcium (Ca^{++}) wave sweeps over the surface; it emanates from the point of sperm entry: we briefly discuss this problem in Section 13.6 below. Chemical concentration waves such as those found with the Belousov-Zhabotinskii reaction are visually dramatic examples (see Chapter 1, Volume II). From the analysis on insect dispersal in Section 11.3 in Chapter 11 we can also expect wave phenomena in that area, and in interacting population models where spatial effects are important. Another example, related to interacting populations, is the progressing wave of an epidemic, of which the rabies epizootic currently spreading across Europe is a dramatic and disturbing example; we study a model for this in some detail in Chapter 13. The movement of microorganisms moving into a food source, chemotactically directed, is another. The slime mould *Dictyostelium discoideum* is a particularly widely studied example of chemotaxis; we discuss this phenomenon later (see the photograph in Figure 1.1, Volume II which shows associated waves).

The book by Winfree (2000) is replete with wave phenomena in biology. The introductory text on mathematical models in molecular and cellular biology edited by Segel (1980) also deals with some aspects of wave motion. Although not so application oriented, there are several books on reaction diffusion equations such as by Fife (1979), Britton (1986) and Grindrod (1996) which are all relevant. Zeeman (1977) considers wave phenomena in development and other biological areas from a catastrophe theory standpoint.

The point to be emphasised is the widespread existence of wave phenomena in the biomedical sciences which necessitates a study of travelling waves in depth and of the modelling and analysis involved. This chapter and Chapter 1, Volume II (with many other examples throughout Volume II) deal with various aspects of wave behaviour where diffusion plays a crucial role. The waves studied here are quite different from those discussed in Chapter 12. The mathematical literature on them is now vast, so the

Insect dispersal

epidemics

number of topics and the depth of the discussions have to be severely limited. Among other things, we shall cover what is now accepted as part of the basic theory in the field and describe two practical problems, one associated with insect dispersal and control and the other related to calcium waves on amphibian eggs.

In developing living systems there is almost continual interchange of information at both the inter- and intra-cellular level. Such communication is necessary for the sequential development and generation of the required pattern and form in, for example, embryogenesis. Propagating waveforms of varying biochemical concentrations are one means of transmitting such biochemical information. In the developing embryo, diffusion coefficients of biological chemicals can be very small; values of the order of 10^{-9} to $10^{-11} \text{ cm}^2 \text{ sec}^{-1}$ are fairly common. Such small diffusion coefficients imply that to cover macroscopic distances of the order of several millimetres requires a very long time if diffusion is the principal process involved. Estimation of diffusion coefficients for insect dispersal in interacting populations is now studied with care and sophistication (see, for example, Kareiva 1983 and Tilman and Kareiva 1998): not surprisingly the values are larger and species-dependent.

With a standard diffusion equation in one space dimension, which from Section 11.1 is typically of the form

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad (13.1)$$

for a chemical of concentration u , the time to convey information in the form of a changed concentration over a distance L is $O(L^2/D)$. You get this order estimate from the equation using dimensional arguments, similarity solutions or more obviously from the classical solution given by equation (11.10) in Chapter 11. So, if L is of the order of 1 mm, typical times with the above diffusion coefficients are $O(10^7$ to 10^9 sec), which is excessively long for most processes in the early stages of embryonic development. Simple diffusion therefore is unlikely to be the main vehicle for transmitting information over significant distances. A possible exception is the generation of butterfly wing patterns, which takes place during the pupal stage and involves several days (for example, Murray 1981 and Nijhout 1991).

←
small
diffusion

←

←

$$t = O\left[\frac{10^{-2}}{10^{-9}}\right]$$

time scale



reaction-diff
eqn.

$$\frac{\partial u}{\partial t} = f(u) + D \frac{\partial^2 u}{\partial x^2}, \quad (13.2)$$

←

where u is the concentration, $f(u)$ represents the kinetics and D is the diffusion coefficient, here taken to be constant.

We must first decide what we mean by a travelling wave. We saw in Chapter 11 that the solutions (11.21) and (11.24) described a kind of wave, where the shape and speed of propagation of the front continually changed. Customarily a travelling wave is taken

to be a wave which travels without change of shape, and this will be our understanding here. So, if a solution $u(x, t)$ represents a travelling wave, the shape of the solution will be the same for all time and the speed of propagation of this shape is a constant, which we denote by c . If we look at this wave in a travelling frame moving at speed c it will appear stationary. A mathematical way of saying this is that if the solution

$$u(x, t) = u(x - ct) = u(z), \quad z = x - ct \quad (13.3)$$

then $u(x, t)$ is a travelling wave, and it moves at constant speed c in the positive x -direction. Clearly if $x - ct$ is constant, so is u . It also means the coordinate system moves with speed c . A wave which moves in the negative x -direction is of the form $u(x + ct)$. The wavespeed c generally has to be determined. The dependent variable z is sometimes called the wave variable. When we look for travelling wave solutions of an equation or system of equations in x and t in the form (13.3), we have $\partial u / \partial t = -cdu / dz$ and $\partial u / \partial x = du / dz$. So partial differential equations in x and t become ordinary differential equations in z . To be physically realistic $u(z)$ has to be bounded for all z and nonnegative with the quantities with which we are concerned, such as chemicals, populations, bacteria and cells.

It is part of the classical theory of linear parabolic equations, such as (13.1), that there are no physically realistic travelling wave solutions. Suppose we look for solutions in the form (13.3); then (13.1) becomes

$$D \frac{d^2 u}{dz^2} + c \frac{du}{dz} = 0 \Rightarrow u(z) = A + Be^{-cz/D},$$

where A and B are integration constants. Since u has to be bounded for all z , B must be zero since the exponential becomes unbounded as $z \rightarrow -\infty$. $u(z) = A$, a constant, is not a wave solution. In marked contrast the parabolic reaction diffusion equation (13.2) can exhibit travelling wave solutions, depending on the form of the reaction/interaction term $f(u)$. This solution behaviour was a major factor in starting the whole mathematical field of reaction diffusion theory.

Although most realistic models of biological interest involve more than one dimension and more than one dependent variable, whether concentration or population, there are several multi-species systems which reasonably reduce to a one-dimensional single-species mechanism which captures key features. This chapter therefore is not simply a pedagogical mathematical exposition of some common techniques and basic theory. We discuss two very practical problems, one in ecology and the other in developmental biology: both belong to important areas where modelling has played a significant role.

1D captur
Key featur
 $N > 1D$
+realistic

13.2 Fisher-Kolmogoroff Equation and Propagating Wave Solutions

Fisher (1937)
Kolmogorov (1937)
precursor work Luther (1)

The classic simplest case of a nonlinear reaction diffusion equation (13.2) is

$$\frac{\partial u}{\partial t} = ku(1-u) + D \frac{\partial^2 u}{\partial x^2}, \quad k>0, \quad D>0 \quad (13.4)$$

where k and D are positive parameters. It was suggested by Fisher (1937) as a deterministic version of a stochastic model for the spatial spread of a favoured gene in a population. It is also the natural extension of the logistic growth population model discussed in Chapter 11 when the population disperses via linear diffusion. This equation and its travelling wave solutions have been widely studied, as has been the more general form with an appropriate class of functions $f(u)$ replacing $ku(1-u)$. The seminal and now classical paper is that by Kolmogoroff et al. (1937). The books by Fife (1979), Britton (1986) and Grindrod (1996) mentioned above give a full discussion of this equation and an extensive bibliography. We discuss this model equation in the following section in some detail, not because in itself it has such wide applicability but because it is the prototype equation which admits travelling wavefront solutions. It is also a convenient equation from which to develop many of the standard techniques for analysing single-species models with diffusive dispersal.

Although (13.4) is now referred to as the Fisher–Kolmogoroff equation, the discovery, investigation and analysis of travelling waves in chemical reactions was first reported by Luther (1906). This rediscovered paper has been translated by Arnold et al. (1987). Luther's paper was first presented at a conference; the discussion at the end of his presentation (and it is included in the Arnold et al. 1988 translation) is very interesting. There, Luther states that the wavespeed is a simple consequence of the differential equations. Showalter and Tyson (1987) put Luther's (1906) remarkable discovery and analysis of chemical waves in a modern context. Luther obtained the wavespeed in terms of parameters associated with the reactions he was studying. The analytical form is the same as that found by Kolmogoroff et al. (1937) and Fisher (1937) for (13.4).

Let us now consider (13.4). It is convenient at the outset to rescale (13.4) by writing

$$t^* = kt, \quad x^* = x \left(\frac{k}{D} \right)^{1/2} \quad (13.5)$$

and, omitting the asterisks for notational simplicity, (13.4) becomes

dimensionless

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}. \quad (13.6)$$

In the spatially homogeneous situation the steady states are $u = 0$ and $u = 1$, which are respectively unstable and stable. This suggests that we should look for travelling wavefront solutions to (13.6) for which $0 \leq u \leq 1$; negative u has no physical meaning with what we have in mind for such models.

If a travelling wave solution exists it can be written in the form (13.3), say

$$u(x, t) = U(z), \quad z = x - ct, \quad (13.7)$$

where c is the wavespeed. We use $U(z)$ rather than $u(z)$ to avoid any nomenclature confusion. Since (13.6) is invariant if $x \rightarrow -x$, c may be negative or positive. To be specific we assume $c \geq 0$. Substituting this travelling waveform into (13.6), $U(z)$ satisfies

PAUSE FOR A QUICK ODE "REFRESH".

① **logistic equation** - logistic growth in population

$$\frac{dN}{dt} = rN(1 - N/K), \quad N(0) = N_0.$$

per capita birth rate is $r(1 - \frac{N}{K})$

slows down for large populations

K = carrying capacity of the environment (in connection with supplies)

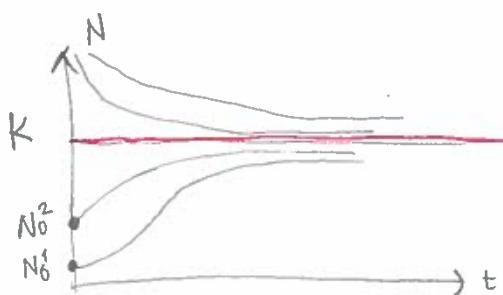
2 steady states (equilibrium states, critical points)

$$N_1=0, \quad N_2=K,$$

N_1 is unstable : linearization about N_1 $\Rightarrow \frac{dN}{dt} \approx rN$

N_2 is stable : linearization about N_2 $\Rightarrow \frac{dN}{dt} \approx -r(N-K)$

Solution : $N(t) = \frac{N_0 K e^{rt}}{K + N_0 (e^{rt} - 1)} \rightarrow K, \text{ as } t \rightarrow \infty.$



Lotka-Volterra - Predator-Prey Model
 (N_2) (N_1)

$$\begin{cases} \frac{dN_1}{dt} = k_1 N_1 - k_2 N_1 N_2 & , N_1(0) = N_1^0 \\ \frac{dN_2}{dt} = k_3 N_2 - k_4 N_2 & , N_2(0) = N_2^0 \end{cases}$$

critical points $(N_1^*, N_2^*) = (0, 0) = \left(\frac{k_4}{k_3}, \frac{k_1}{k_2}\right)$

$$N_1(t) = N_1^* + \varepsilon \tilde{N}_1(t)$$

$$N_2(t) = N_2^* + \varepsilon \tilde{N}_2(t)$$

$$\varepsilon \ll 1$$

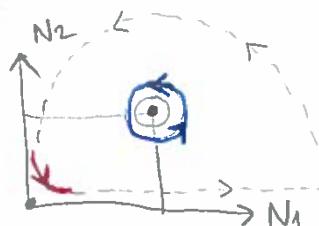
$O(\varepsilon)$ -equations

$$\frac{d}{dt} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} = \begin{bmatrix} (k_1 - k_2 N_2^*) & -k_2 N_1^* \\ k_3 N_2^* & (k_3 N_1^* - k_4) \end{bmatrix} \begin{bmatrix} \tilde{N}_1(t) \\ \tilde{N}_2(t) \end{bmatrix}$$

When $(N_1^*, N_2^*) = (0, 0)$, $\lambda_1 = k_1, \lambda_2 = -k_4$

when $(N_1^*, N_2^*) = \left(\frac{k_4}{k_3}, \frac{k_1}{k_2}\right)$

$$\lambda_{1,2} = \pm i \sqrt{k_1 k_4}$$



$$u_t = u(1-u) + u_{xx}$$

$$u(x,t) = U(z)$$

$$z = x - ct$$

$$U'' + cU' + U(1-U) = 0, \quad (13.8)$$

where primes denote differentiation with respect to z . A typical wavefront solution is where U at one end, say, as $z \rightarrow -\infty$, is at one steady state and as $z \rightarrow \infty$ it is at the other. So here we have an eigenvalue problem to determine the value, or values, of c such that a nonnegative solution U of (13.8) exists which satisfies

$$\lim_{z \rightarrow \infty} U(z) = 0, \quad \lim_{z \rightarrow -\infty} U(z) = 1.$$

At this stage we do not address the problem of how such a travelling wave solution might evolve from the partial differential equation (13.6) with given initial conditions $u(x, 0)$; we come back to this point later.

We study (13.8) for U in the (U, V) phase plane where

$$U' = V, \quad V' = -cV - U(1-U), \quad (13.10)$$

which gives the phase plane trajectories as solutions of

$$\frac{dV}{dU} = \frac{-cV - U(1-U)}{V}.$$

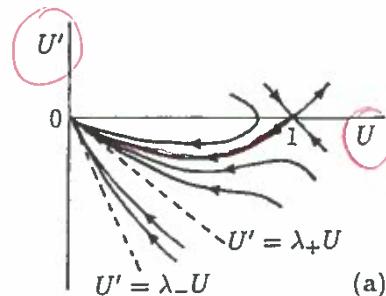
This has two singular points for (U, V) , namely, $(0, 0)$ and $(1, 0)$: these are the steady states of course. A linear stability analysis (see Appendix A) shows that the eigenvalues λ for the singular points are

$$(0, 0): \quad \lambda_{\pm} = \frac{1}{2} [-c \pm (c^2 - 4)^{1/2}] \Rightarrow \begin{cases} \text{stable node} & \text{if } c^2 > 4 \\ \text{stable spiral} & \text{if } c^2 < 4 \end{cases} \quad \begin{matrix} \text{real e-values} \\ \text{complex conj} \end{matrix} \quad (13.12)$$

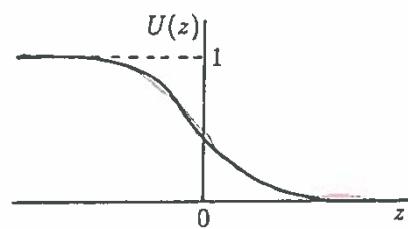
$$(1, 0): \quad \lambda_{\pm} = \frac{1}{2} [-c \pm (c^2 + 4)^{1/2}] \Rightarrow \text{saddle point.} \quad \lambda_+ > 0, \lambda_- < 0$$

Figure 13.1(a) illustrates the phase plane trajectories.

If $c \geq c_{\min} = 2$ we see from (13.12) that the origin is a stable node, the case when $c = c_{\min}$ giving a degenerate node. If $c^2 < 4$ it is a stable spiral; that is, in the vicinity



(a)



(b)

Figure 13.1. (a) Phase plane trajectories for equation (13.8) for the travelling wavefront solution: here $c^2 > 4$. (b) Travelling wavefront solution for the Fisher-Kolmogoroff equation (13.6): the wave velocity $c \geq 2$.

Some additional information

16:

of the origin U oscillates. By continuity arguments, or simply by heuristic reasoning from the phase plane sketch of the trajectories in Figure 13.1(a), there is a trajectory from $(1, 0)$ to $(0, 0)$ lying entirely in the quadrant $U \geq 0, U' \leq 0$ with $0 \leq U \leq 1$ for all wavespeeds $c \geq c_{\min} = 2$. In terms of the original dimensional equation (13.4), the range of wavespeeds satisfies

$$c \geq c_{\min} = 2(kD)^{1/2}. \quad (13.13)$$

Figure 13.1(b) is a sketch of a typical travelling wave solution. There are travelling wave solutions for $c < 2$ but they are physically unrealistic since $U < 0$, for some z , because in this case U spirals around the origin. In these, $U \rightarrow 0$ at the leading edge with decreasing oscillations about $U = 0$.

A key question at this stage is what kind of initial conditions $u(x, 0)$ for the original Fisher–Kolmogoroff equation (13.6) will evolve to a travelling wave solution and, if such a solution exists, what is its wavespeed c ? This problem and its generalisations have been widely studied analytically; see the references in the books cited above in Section 13.1. Kolmogoroff et al. (1937) proved that if $u(x, 0)$ has compact support, that is,

$$u(x, 0) = u_0(x) \geq 0, \quad u_0(x) = \begin{cases} 1 & \text{if } x \leq x_1 \\ 0 & \text{if } x \geq x_2 \end{cases} \quad (13.14)$$

where $x_1 < x_2$ and $u_0(x)$ is continuous in $x_1 < x < x_2$, then the solution $u(x, t)$ of (13.6) evolves to a travelling wavefront solution $U(z)$ with $z = x - 2t$. That is, it evolves to the wave solution with minimum speed $c_{\min} = 2$. For initial data other than (13.14) the solution depends critically on the behaviour of $u(x, 0)$ as $x \rightarrow \pm\infty$.

The dependence of the wavespeed c on the initial conditions at infinity can be seen easily from the following simple analysis suggested by Mollison (1977). Consider first the leading edge of the evolving wave where, since u is small, we can neglect u^2 in comparison with u . Equation (13.6) is linearised to

$$u_t = u - u^2 + u_{xx} \quad \frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}. \quad (13.15)$$

Consider now

$$u(x, 0) \sim Ae^{-ax} \quad \text{as } x \rightarrow \infty, \quad (13.16)$$

where $a > 0$ and $A > 0$ is arbitrary, and look for travelling wave solutions of (13.15) in the form

$$u(x, t) = Ae^{-a(x-ct)}. \quad (13.17)$$

near wavefront approx.

We think of (13.17) as the leading edge form of the wavefront solution of the nonlinear equation. Substitution of the last expression into the linear equation (13.15) gives the dispersion relation, that is, a relationship between c and a ,

$$\frac{\omega}{k}$$

11.3 Models for Animal Dispersal

Diffusion models form a reasonable basis for studying insect and animal dispersal and invasion; this and other aspects of animal population models are discussed in detail, for example, by Okubo (1980, 1986), Shigesada (1980) and Lewis (1997). Dispersal of interacting species is discussed by Shigesada et al. (1979) and of competing species by Shigesada and Roughgarden (1982). Kareiva (1983) has shown that many species appear to disperse according to a reaction diffusion model with a constant diffusion coefficient. He gives actual values for the diffusion coefficients which he obtained from experiments on a variety of insect species. Kot et al. (1996) studied dispersal of organisms in general and importantly incorporated real data (see also Kot 2001). A common feature of insect populations is their discrete time population growth. As would be expected intuitively this can have a major effect on their spatial dispersal. The model equations involve the coupling of discrete time with continuous space, a topic investigated by Kot (1992) and Neubert et al. (1995). The book of articles edited by Tilman and Kareiva (1998) is a useful sourcebook for the role of space in this general area. The articles address, for example, the question of persistence of endangered species, biodiversity, disease dynamics, multi-species competition and so on. The books by Renshaw (1991) and Williamson (1996) are other very good texts for the study of species invasion phenomena: these books have numerous examples. The excellent, more mathematical and modelling oriented, book by Shigesada and Kawasaki (1997) discusses biological invasions of mammals, birds, insects and plants in various forms, of which diffusion is just one mechanism. For anyone seriously interested in modelling these phenomena these books are required reading.

One extension of the classical diffusion model which is of particular relevance to insect dispersal is when there is an increase in diffusion due to population pressure. One such model has the diffusion coefficient, or rather the flux J , depending on the population density n such that D increases with n ; that is,

population flux
depending on
population density n

$$J = -D(n)\nabla n, \quad \frac{dD}{dn} > 0.$$

Darcy's Law (flow in porous medium) $q = -\frac{k \nabla p}{\mu}$
 (11.19)
 $q = \text{flux, discharge/area}$
 $k = \text{permeability}$

A typical form for $D(n)$ is $D_0(n/n_0)^m$, where $m > 0$ and D_0 and n_0 are positive constants. The dispersal equation for n without any growth term is then

$$\frac{\partial n}{\partial t} = D_0 \nabla \cdot \left[\left(\frac{n}{n_0} \right)^m \nabla n \right].$$

In one dimension

1D.
 $\frac{\partial n}{\partial t} = D_0 \frac{\partial}{\partial x} \left[\left(\frac{n}{n_0} \right)^m \frac{\partial n}{\partial x} \right], \quad (11.20)$

which has an exact analytical solution of the form

diffusion equation
with variable diffusion coefficient

Analytical solution

$$n(x, t) = \frac{n_0}{\lambda(t)} \left[1 - \left\{ \frac{x}{r_0 \lambda(t)} \right\}^2 \right]^{1/m}, \quad |x| \leq r_0 \lambda(t)$$

$$= 0, \quad |x| > r_0 \lambda(t), \quad (11.21)$$

compact support

where

$\lambda(t) = \left(\frac{t}{t_0} \right)^{1/(2+m)},$	$r_0 = \frac{Q \Gamma(\frac{1}{m} + \frac{3}{2})}{\{\pi^{1/2} n_0 \Gamma(\frac{1}{m} + 1)\}},$
$t_0 = \frac{r_0^2 m}{2 D_0 (m + 2)},$	

$$(11.22)$$

where Γ is the gamma function and Q is the initial number of insects released at the origin. It is straightforward to check that (11.21) is a solution of (11.20) for all r_0 . The evaluation of r_0 comes from requiring the integral of n over all x to be equal to Q . (In another context (11.20) is known as the porous media equation.) The population is identically zero for $x > r_0 \lambda(t)$. This solution is fundamentally different from that when $m = 0$, namely, (11.10). The difference is due to the fact that $D(0) = 0$. The solution represents a kind of wave with the front at $x = x_f = r_0 \lambda(t)$. The derivative of n is discontinuous here. The wave 'front,' which we define here as the point where $n = 0$, propagates with a speed $dx_f/dt = r_0 d\lambda/dt$, which, from (11.22), decreases with time for all m . The solution for n is illustrated schematically in Figure 11.2. The dispersal patterns for grasshoppers exhibit a similar behaviour to this model (Aikman and Hewitt 1972). Without any source term the population n , from (11.21), tends to zero as $t \rightarrow \infty$. Shigesada (1980) proposed such a model for animal dispersal in which she took the linear diffusion dependence $D(n) \propto n$; see also Shigesada and Kawasaki (1997).

The equivalent plane radially symmetric problem with Q insects released at $r = 0$ at $t = 0$ satisfies the equation

$$\frac{\partial n}{\partial t} = \left(\frac{D_0}{r} \right) \frac{\partial}{\partial r} \left[r \left(\frac{n}{n_0} \right)^m \frac{\partial n}{\partial r} \right] \quad (11.23)$$

with solution

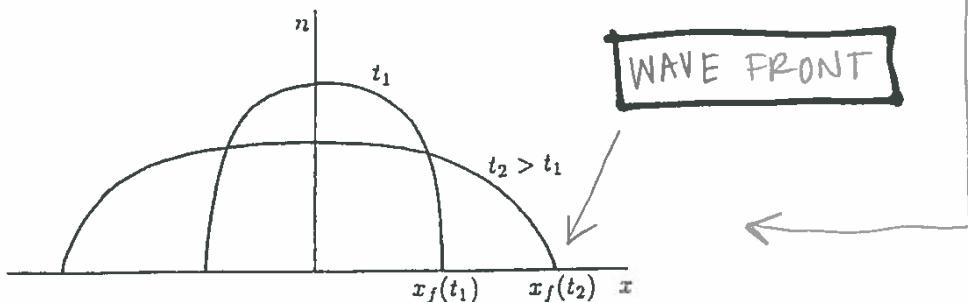


Figure 11.2. Schematic solution, from (11.21), of equation (11.20) as a function of x at different times t . Note the discontinuous derivative at the wavefront $x_f(t) = r_0 \lambda(t)$.

fact that the waves are stable to finite domain perturbations makes it clear why typical numerical simulations of the Fisher–Kolmogoroff equation result in stable wavefront solutions with speed $c = 2$.

A natural extension of (11.20):

13.4 Density-Dependent Diffusion-Reaction Diffusion Models and Some Exact Solutions

We saw in Section 11.3 in Chapter 11 that in certain insect dispersal models the diffusion coefficient D depended on the population u . There we did not include any growth dynamics. If we wish to consider longer timescales then we should include such growth terms in the model. A natural extension to incorporate density-dependent diffusion is thus, in the one-dimensional situation, to consider equations of the form

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial}{\partial x} \left[D(u) \frac{\partial u}{\partial x} \right], \quad (13.39)$$

where typically $D(u) = D_0 u^m$, with D_0 and m positive constants. Here we consider functions $f(u)$ which have two zeros, one at $u = 0$ and the other at $u = 1$. Equations in which $f \equiv 0$ have been studied much more widely than those with nonzero f ; see, for example, Chapter 11. To be even more specific we consider $f(u) = ku^p(1 - u^q)$, where p and q are positive constants. By a suitable rescaling of t and x we can absorb the parameters k and D_0 and the equations we thus consider in this section are then of the general form

$$\frac{\partial u}{\partial t} = u^p(1 - u^q) + \frac{\partial}{\partial x} \left[u^m \frac{\partial u}{\partial x} \right], \quad (13.40)$$

where p , q and m are positive parameters. If we write out the diffusion term in full we get

$$\frac{\partial u}{\partial t} = u^p(1 - u^q) + mu^{m-1} \left(\frac{\partial u}{\partial x} \right)^2 + u^m \frac{\partial^2 u}{\partial x^2}$$

which shows that the nonlinear diffusion can be thought of as contributing an equivalent convection with ‘velocity’ $-mu^{m-1}\partial u/\partial x$.

It might be argued that the forms in (13.40) are rather special. However with the considerable latitude to choose p , q and m such forms can qualitatively mimic more complicated forms for which only numerical solutions are possible. The usefulness of analytical solutions, of course, is the ease with which we can see how solutions depend analytically on the parameters. In this way we can then infer the qualitative behaviour of the solutions of more complicated but more realistic model equations. There are, however, often hidden serious pitfalls, one of which is important and which we point out below.

CHAPTER
ELEVEN

MULTIPLE-SCALE ANALYSIS



And here—ah, now, this really is something a little recherché.

—Sherlock Holmes, *The Musgrave Ritual*
Sir Arthur Conan Doyle

(E) **11.1 RESONANCE AND SECULAR BEHAVIOR**



Multiple-scale analysis is a very general collection of perturbation techniques that embodies the ideas of both boundary-layer theory and WKB theory. Multiple-scale analysis is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems.

In this section we show how nonuniformity can appear in a regular perturbation expansion as a result of resonant interactions between consecutive orders of perturbation theory. To illustrate, we examine a simple perturbation problem, show how resonances occur and lead to a nonuniformly valid perturbation expansion, and finally show how to interpret and eliminate these nonuniformities. The formal development of multiple-scale analysis is postponed to Sec. 11.2.

Resonance

The phenomenon of resonance is nicely illustrated by the differential equation

$$\frac{d^2}{dt^2} \ddot{y}(t) + y(t) = \cos(\omega t). \quad (11.1.1)$$

This equation represents a harmonic oscillator of natural frequency 1 which is driven by a periodic external force of frequency ω . The general solution to this equation for $|\omega| \neq 1$ has the form

$$y(t) = A \cos t + B \sin t + \frac{\cos(\omega t)}{1 - \omega^2}, \quad |\omega| \neq 1. \quad (11.1.2)$$

Observe that for all $|\omega| \neq 1$ the solution remains bounded for all t . If $|\omega|$ is close to 1, the amplitude of oscillation becomes large because the system absorbs large amounts of energy from the external force. Nevertheless, the amplitude of the system is still bounded when $|\omega| \neq 1$ because the system is oscillating out of phase with the driving force.

The solution in (11.1.2) is incorrect when $|\omega| = 1$. The correct solution has an amplitude which grows with t :

$$y(t) = A \cos t + B \sin t + \frac{1}{2}t \sin t, \quad |\omega| = 1. \quad (11.1.3)$$

The amplitude of oscillation of this solution is unbounded as $t \rightarrow \infty$ because the oscillator continually absorbs energy from the periodic external force. This system is in *resonance* with the external force.

The term $\frac{1}{2}t \sin t$, whose amplitude grows with t , is said to be a *secular* term. The secular term $\frac{1}{2}t \sin t$ has appeared because the inhomogeneity $\cos t$ in (11.1.1) with $|\omega| = 1$ is itself a solution of the homogeneous equation associated with (11.1.1): $d^2y/dt^2 + y = 0$. In general, secular terms always appear whenever the inhomogeneous term is itself a solution of the associated homogeneous *constant-coefficient* differential equation. A secular term always grows more rapidly than the corresponding solution of the homogeneous equation by at least a factor of t .

Example 1 Appearance of secular terms.

- (a) The solution to the differential equation $d^2y/dt^2 - y = e^{-t}$ has a secular term because e^{-t} satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^{-t} + Be^t - \frac{1}{2}te^{-t}$. The particular solution $-\frac{1}{2}te^{-t}$ is secular relative to the homogeneous solution Ae^{-t} ; we must regard the term $-\frac{1}{2}te^{-t}$ as secular even though it is negligible compared with the homogeneous solution Be^t as $t \rightarrow \infty$.
- (b) The solution to the differential equation $d^2y/dt^2 - 2dy/dt + y = e^t$ has a secular term because e^t satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^t + Be^t + \frac{1}{2}t^2e^t$. In this case, the particular solution $\frac{1}{2}t^2e^t$ is secular with respect to all solutions of the associated homogeneous equation.

Nonuniformity of Regular Perturbation Expansions

The appearance of secular terms signals the nonuniform validity of perturbation expansions for large t . The nonlinear oscillator equation (Duffing's equation)

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad (11.1.4)$$

provides a good illustration of what we mean by nonuniformity. A perturbative solution of this equation is obtained by expanding $y(t)$ as a power series in ϵ :

$$y(t) = \sum_{n=0}^{\infty} \epsilon^n y_n(t), \quad (11.1.5)$$

where $y_0(0) = 1, y'_0(0) = 0, y_n(0) = y'_n(0) = 0$ ($n \geq 1$). Substituting (11.1.5) into the differential equation (11.1.4) and equating coefficients of like powers of ϵ gives a sequence of linear differential equations of which all but the first are inhomogeneous:

$$\mathcal{O}(\epsilon^0) \quad y''_0 + y_0 = 0, \quad (11.1.6a)$$

$$\mathcal{O}(\epsilon^1) \quad y''_1 + y_1 = -y_0^3, \quad (11.1.6b)$$

and so on.

The solution to (11.1.6a) which satisfies $y_0(0) = 1, y'_0(0) = 0$ is

$$y_0(t) = \cos t.$$

To solve (11.1.6b) we invoke the trigonometric identity $\cos^3 t = \frac{1}{4} \cos 3t + \frac{3}{4} \cos t$ to rewrite the inhomogeneous term. The formulas in (11.1.2)-(11.1.3) then provide the general solution to (11.1.6b):

$$y_1(t) = A \cos t + B \sin t + \frac{1}{32} \cos 3t - \frac{3}{8} t \sin t;$$

the particular solution satisfying $y_1(0) = y'_1(0) = 0$ is

$$y_1(t) = \frac{1}{32} \cos 3t - \frac{3}{8} t \sin t.$$

We observe that $y_1(t)$ contains a secular term. This secularity necessarily occurs because $\cos^3 t$ contains a component, $\frac{3}{4} \cos t$, whose frequency equals the natural frequency of the unperturbed oscillator.

In summary, the first-order perturbative solution to (11.1.4) is

$$y(t) = \cos t + \varepsilon \left[\frac{1}{32} \cos 3t - \frac{3}{8} t \sin t \right] + O(\varepsilon^2), \quad \varepsilon \rightarrow 0+. \quad (11.1.7)$$

We emphasize that the term $O(\varepsilon^2)$ in the above expression means that for fixed t the error between $y(t)$ and $y_0(t) + \varepsilon y_1(t)$ is at most of order ε^2 as $\varepsilon \rightarrow 0+$. The nonuniformity of this result surfaces if we consider large values of t —specifically, values of t of order $1/\varepsilon$ or larger as $\varepsilon \rightarrow 0+$. For such large values of t , the secular term in $y_1(t)$ suggests that the amplitude of oscillation grows with t . However, as we will now show, the exact solution $y(t)$ remains bounded for all t .

bounded solution

y_1 larger than y_0
@ $t = O(1/\varepsilon)$
ordering can be violated

Boundedness of the Solution to (11.1.4)

To show that the solution to (11.1.4) is bounded for all t , we construct an integral of the differential equation. Multiplying (11.1.4) by the integrating factor dy/dt converts each term in the differential equation to an exact derivative:

$$\frac{d}{dt} \left[\frac{1}{2} \left(\frac{dy}{dt} \right)^2 + \frac{1}{2} y^2 + \frac{1}{4} \varepsilon y^4 \right] = 0.$$

Thus,

$$\frac{1}{2} \left(\frac{dy}{dt} \right)^2 + \frac{1}{2} y^2 + \frac{1}{4} \varepsilon y^4 = C, \quad (11.1.8)$$

where C is a constant. Since $y(0) = 1$ and $y'(0) = 0$, $C = \frac{1}{2} + \frac{1}{4}\varepsilon$. When $\varepsilon > 0$, the integral in (11.1.8) shows that $\frac{1}{2}y^2 \leq C$ for all t . Therefore, $|y(t)|$ is bounded for all t by $\sqrt{1 + \varepsilon/2}$.

The argument just given is frequently used in applied mathematics to prove boundedness of solutions to both ordinary and partial differential equations. The integral in (11.1.8) is called an energy integral. Equation (11.1.8) may be interpreted graphically as a closed bounded orbit in the phase plane whose axes are labeled by y and dy/dt (see Fig. 11.1).

Perturbative Construction of a Bounded Solution to (11.1.4)

We have arrived at an apparent paradox: we have shown that the exact solution $y(t)$ to (11.1.4) is bounded for all t but that the first-order perturbative solution in (11.1.7) is secular (grows with t for large t). The resolution of this paradox lies in

$\cos 3t + \frac{3}{2} \cos t$
L3) then provide

necessarily occurs
nals the natural

is
 $\rightarrow 0 \pm$ (11.1.7)

that for fixed t
is $\varepsilon \rightarrow 0 \pm$. The
specifically,
of t , the secular
 ε . However, as

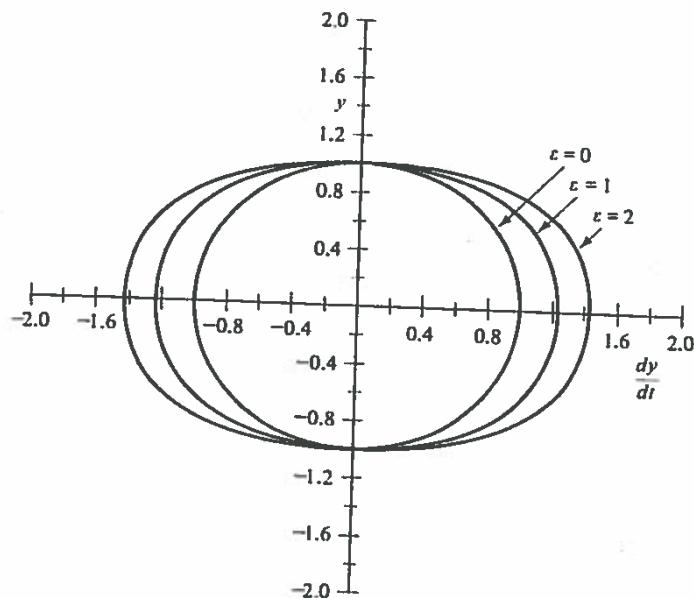


Figure 11.1 A phase-plane plot (y versus dy/dt) of solutions to Duffing's equation $d^2y/dt^2 + y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0, 1$, and 2 . The orbits shown are constant-energy curves [see (11.1.8)] which satisfy $(dy/dt)^2 + y^2 + \varepsilon y^4/2 = 1 + \varepsilon/2$.

the summation of the perturbation series (11.1.5). We know that the problem (11.1.4) is a regular perturbation problem as $\varepsilon \rightarrow 0+$ for fixed t (see Sec. 7.2). Therefore, the series (11.1.5) converges to the solution $y(t)$ for each t . We conclude that although order by order each term in the perturbation expansion may be secular, the secularity must disappear when the series is summed.

To illustrate how summing a perturbation series can eliminate secularity, consider the perturbation series

$$1 - rt + \frac{1}{2}\varepsilon^2 t^2 - \frac{1}{6}\varepsilon^3 t^3 + \cdots + \varepsilon^n t^n [(-1)^n/n!] + \cdots, \quad \varepsilon \rightarrow 0+.$$

Each term in this series is secular when t is of order $1/\varepsilon$ or larger. Nevertheless, the sum of the series $e^{-\alpha}$ is bounded for all positive t !

We will now examine the more complicated perturbation series (11.1.5) and show that the sum of the most secular terms in each order in perturbation theory is actually not secular. We will show, using an inductive argument, that the most secular term in $y_n(t)$ has the form

$$A_n t^n e^{it} + A_n^* t^n e^{-it}, \quad (11.1.9)$$

where $*$ denotes complex conjugation. There are less secular terms in $y_n(t)$ which grow like t^k ($k < n$), but we ignore such terms for now.

The final result of our calculations will be

$$A_n = \frac{1}{2} \frac{1}{n!} \left(\frac{3i}{8}\right)^n. \quad (11.1.10)$$

SUMMING most secular terms in $y_n(t)$, gives

Using this formula for A_n we see that the sum of the most secular terms in the perturbation series (11.1.5) is a cosine function:

$$\sum_{n=0}^{\infty} \frac{1}{2} \varepsilon^n t^n \left[\frac{1}{n!} \left(\frac{3i}{8} \right)^n e^{it} + \frac{1}{n!} \left(-\frac{3i}{8} \right)^n e^{-it} \right] = \cos \left[t \left(1 + \frac{3}{8} \varepsilon \right) \right]. \quad (11.1.11)$$

Observe that this expression is not secular; it remains bounded for all t .

The expression (11.1.11) is a much better approximation to the exact solution $y(t)$ than $y_0(t) = \cos t$ because it is a good approximation to $y(t)$ for $0 \leq t = O(1/\varepsilon)$. The difference between $y(t)$ and $\cos t$ is small so long as $0 \leq t \ll 1/\varepsilon$ ($\varepsilon \rightarrow 0+$), while $\cos [t(1 + \frac{3}{8}\varepsilon)]$ is an accurate approximation to $y(t)$ over a much larger range of t . These assertions are explained as follows. In order that $y_0(t)$ be a good approximation to $y(t)$, it is necessary that $\varepsilon^n y_n(t) \ll y_0(t)$ ($\varepsilon \rightarrow 0+$) for all $n \geq 1$; this is true if $0 \leq t \ll 1/\varepsilon$. On the other hand, the terms that we ignored in deriving (11.1.11) all have the form

$$\varepsilon [A \varepsilon^k (\varepsilon t)^l e^{imt} + A^* \varepsilon^k (\varepsilon t)^l e^{-imt}],$$

where k, l, m are nonnegative integers. Therefore, when $t = O(1/\varepsilon)$, each of these ignored terms is in fact negligible compared to at least one of the secular terms included in (11.1.11). We accept without proof the nontrivial result that the sum of all these small terms is still small. The higher-order terms are analyzed in Probs. 11.5 to 11.7.

We interpret the formula in (11.1.11) to mean that the cubic anharmonic term in (11.1.4) causes a shift in the frequency of the harmonic oscillator $y'' + y = 0$ from 1 to $1 + \frac{3}{8}\varepsilon$. This small frequency shift causes a phase shift which becomes noticeable when t is of order $1/\varepsilon$ (see Figs. 11.2 to 11.4 in Sec. 11.2).

Inductive Derivation of (11.1.10)

Comparing the first-order perturbation theory result in (11.1.7) with (11.1.9) verifies that the coefficient of the most secular terms in zeroth and first order are given correctly by (11.1.10). To establish (11.1.10) for all n , we proceed inductively. The $(n+2)$ th equation in the sequence of equations (11.1.6) determines $y_{n+1}(t)$:

$$y''_{n+1} + y_{n+1} = -I_{n+1}, \quad (11.1.12)$$

where the inhomogeneity I_{n+1} is the coefficient of ε^n in the expansion of $[\sum_{j=0}^{\infty} \varepsilon^j y_j(t)]^3$. Thus,

$$I_{n+1} = \sum_{j+k+l=n} y_j y_k y_l. \quad (11.1.13)$$

The most secular term in $y_{n+1}(t)$ is generated by the most secular terms in $y_j(t)$ for $0 \leq j \leq n$ (see Prob. 11.2). If we assume that (11.1.10) is valid for $A_0, A_1, A_2, \dots, A_n$, then the coefficient of $t^n e^{it}$ in I_{n+1} is given by

$$\frac{1}{8} \left(\frac{3i}{8} \right)^n \sum_{j+k+l=n} \frac{j^{j+k-l} + i^{j+l-k} + i^{k+l-j}}{j! k! l!} = \frac{1}{8} \left(\frac{3i}{8} \right)^n \sum_{j+k+l=n} \frac{(-1)^j + (-1)^k + (-1)^l}{j! k! l!}.$$

The sum in the above expression is just three times the coefficient of x^n in the Taylor expansion of $e^x e^{-x} e^{-x}$ (see Prob. 11.3); therefore, it has the value $3/n!$. Thus, the terms in I_{n+1} which generate the most secular terms in $y_{n+1}(t)$ are

$$\frac{3}{8}(\frac{3}{8}t)^n [i^n e^{it} + (-i)^{n+1} e^{-it}] / n!$$

Substituting these terms into the right side of (11.1.12) and solving for $y_{n+1}(t)$ gives

$$y_{n+1}(t) = \frac{3}{8}(\frac{3}{8}t)^{n+1} [i^{n+1} e^{it} + (-i)^{n+2} e^{-it}] / (n+1)! + \text{less secular terms.}$$

By induction, we conclude that since (11.1.10) is true for $n = 0$, it remains true for all n .

(E) 11.2 MULTIPLE-SCALE ANALYSIS

In Sec. 11.1 we showed how to eliminate the most secular contributions to perturbation theory by simply summing them to all orders in powers of ϵ . The method we used works well but requires a lengthy calculation which can be avoided by using the methods of multiple-scale analysis that are introduced in this section.

Once again, we consider the nonlinear oscillator problem in (11.1.4):

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad y'(0) = 0. \quad (11.2.1)$$

The principal result of the last section is that when t is of order $1/\epsilon$, perturbation theory in powers of ϵ is invalid. Secular terms appear in all orders (except zeroth order) and violate the boundedness of the solution $y(t)$.

A shortcut for eliminating the most secular terms to all orders begins by introducing a new variable $\tau = et$. τ defines a long time scale because τ is not negligible when t is of order $1/\epsilon$ or larger. Even though the exact solution $y(t)$ is a function of t alone, multiple-scale analysis seeks solutions which are functions of both variables t and τ treated as independent variables. We emphasize that expressing y as a function of two variables is an artifice to remove secular effects; the actual solution has t and τ related by $\tau = et$ so that t and τ are ultimately not independent.

The formal procedure consists of assuming a perturbation expansion of the form

$$y(t) = Y_0(t, \tau) + \epsilon Y_1(t, \tau) + \dots \quad (11.2.2)$$

We use the chain rule for partial differentiation to compute derivatives of $y(t)$:

$$\frac{dy}{dt} = \left(\frac{\partial Y_0}{\partial t} + \frac{\partial Y_0}{\partial \tau} \frac{d\tau}{dt} \right) + \epsilon \left(\frac{\partial Y_1}{\partial t} + \frac{\partial Y_1}{\partial \tau} \frac{d\tau}{dt} \right) + \dots$$

However, since $\tau = et$, $d\tau/dt = \epsilon$. Thus,

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial t} + \epsilon \left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t} \right) + O(\epsilon^2). \quad (11.2.3)$$

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Also, differentiating with respect to t again gives

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon \left(2 \frac{\partial^2 Y_0}{\partial \tau \partial t} + \frac{\partial^2 Y_1}{\partial t^2} \right) + O(\varepsilon^2). \quad (11.2.4)$$

Substituting (11.2.4) into (11.2.1) and collecting powers of ε gives

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad (11.2.5)$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -Y_0^3 - 2 \frac{\partial^2 Y_0}{\partial \tau \partial t}. \quad (11.2.6)$$

The most general real solution to (11.2.5) is

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}, \quad (11.2.7)$$

where $A(\tau)$ is an arbitrary complex function of τ .

$A(\tau)$ will be determined by the condition that secular terms do not appear in the solution to (11.2.6). From (11.2.7), the right side of (11.2.6) is

$$e^{it} \left[-3A^2 A^* - 2i \frac{dA}{d\tau} \right] + e^{-it} \left[-3A(A^*)^2 + 2i \frac{dA^*}{d\tau} \right] - e^{3it} A^3 - e^{-3it} (A^*)^3.$$

Note that e^{it} and e^{-it} are solutions of the homogeneous equation $\partial^2 Y_1 / \partial t^2 + Y_1 = 0$. Therefore, if the coefficients of e^{it} and e^{-it} on the right side of (11.2.6) are nonzero, then the solution $Y_1(t, \tau)$ will be secular in t . To preclude the appearance of secularity, we require that the as yet arbitrary function $A(\tau)$ satisfy

$$-3A^2 A^* - 2i \frac{dA}{d\tau} = 0, \quad (11.2.8)$$

$$-3A(A^*)^2 + 2i \frac{dA^*}{d\tau} = 0. \quad (11.2.9)$$

These two complex equations do not overdetermine $A(\tau)$ because they are redundant: one is the complex conjugate of the other. If (11.2.8) and (11.2.9) are satisfied, no secularity appears in (11.2.6), at least through terms of order ε .

To solve (11.2.8) for $A(\tau)$, we represent $A(\tau)$ in polar coordinate form:

$$A(\tau) = R(\tau)e^{i\theta(\tau)}, \quad (11.2.10)$$

where R and θ are real. Substituting into (11.2.8) and equating real and imaginary parts gives

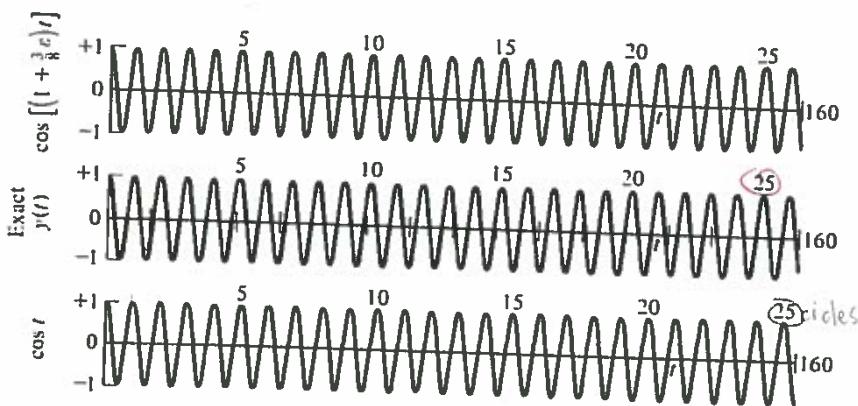
$$\frac{dR}{d\tau} = 0, \quad (11.2.11a)$$

$$\frac{d\theta}{d\tau} = \frac{3}{2} R^2. \quad (11.2.11b)$$

Therefore,

$$A(\tau) = R(0)e^{i\theta(0) + 3iR^2(0)\tau/2} \quad (11.2.12)$$

(11.2.4)



(11.2.5)

(11.2.6)

(11.2.7)

Figure 11.2 The exact solution $y(t)$ to Duffing's equation $d^2y/dt^2 + y + \epsilon t^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\epsilon = 0.1$ (middle graph) compared with perturbative approximations to $y(t)$ (upper and lower graphs). The lower graph is a plot of $\cos t$, the first term in the regular perturbation series for $y(t)$, and the upper graph is a plot of $\cos [(1 + 3\epsilon/8)t]$, the leading-order approximation to $y(t)$ obtained from terms of order ϵ , but $\cos t$ is not valid for large values of t ; when $t = 160$, $\cos t$ is a full cycle out of phase with $y(t)$. The multiple-scale approximation closely approximates $y(t)$, even for large values of t .

and the zeroth-order solution (11.2.7) is

$$Y_0(t, \tau) = 2R(0) \cos [\theta(0) + \frac{3}{2}R^2(0)\tau + t]. \quad (11.2.13)$$

The initial conditions $y(0) = 1$, $y'(0) = 0$ determine $R(0)$ and $\theta(0)$. The condition $y(0) = 1$ becomes $Y_0(0, 0) = 1$, $Y_1(0, 0) = 0$, From (11.2.3), $y'(0) = 0$ becomes $(\partial Y_0/\partial t)(0, 0) = 0$, $(\partial Y_1/\partial t)(0, 0) = -(\partial Y_0/\partial \tau)(0, 0)$, In order to satisfy these conditions, we must choose $R(0) = \frac{1}{2}$ and $\theta(0) = 0$. Therefore, the zeroth-order solution is $Y_0(t, \tau) = \cos [t + \frac{3}{8}\tau]$. Finally, since $\tau = \epsilon t$,

$$y(t) = \cos [t(1 + \frac{3}{8}\epsilon)] + O(\epsilon), \quad \epsilon \rightarrow 0+, \epsilon t = O(1), \quad (11.2.14)$$

and we have reproduced (11.1.11). In Figs. 11.2 to 11.4 we compare the exact solution to (11.2.1) with the approximation in (11.2.14).

A higher-order treatment of (11.2.1) is not completely straightforward. When more than two time scales are employed, there is so much freedom in the perturbation series representation that ambiguities can result (see Probs. 11.5 to 11.7).

(I) 11.3 EXAMPLES OF MULTIPLE-SCALE ANALYSIS

In this section we illustrate the formal multiple-scale technique that was developed in Sec. 11.2 by showing how to solve four elementary examples. The third and fourth of these examples are especially interesting because they show

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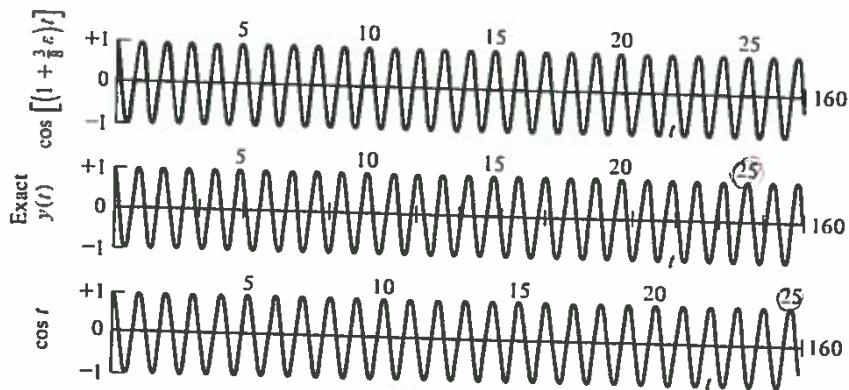
 $\varepsilon = 0.2$ 

Figure 11.3 Same as in Fig. 11.2 but with $c = 0.2$. Note that $\cos t$ is two cycles out of phase with $y(t)$ when $t = 160$.

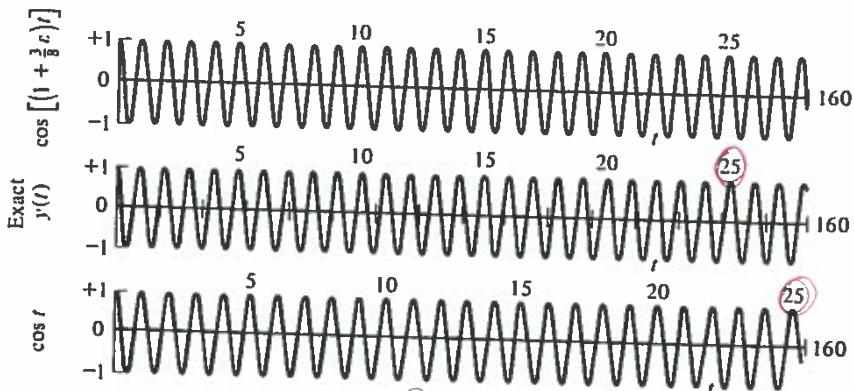
 $\varepsilon = 0.3$ 

Figure 11.4 Same as in Fig. 11.2 but with $\varepsilon = 0.3$. Note that $\cos t$ is three cycles out of phase with $y(t)$ when $t = 160$.

how multiple-scale analysis can reproduce the results of boundary-layer and WKB analysis.

Example 1 *Multiple-scale analysis of a damped oscillator.* Let us consider an harmonic oscillator with a cubic damping term:

$$y'' + y + \varepsilon(y')^3 = 0, \quad y(0) = 1, y'(0) = 0. \quad (11.3.1)$$

If $\varepsilon > 0$, the solution $y(t)$ must decay to 0 as $t \rightarrow \infty$. To prove this assertion, we multiply (11.3.1) by y' and construct an energy integral similar to that in (11.1.8):

$$\frac{d}{dt} \left[\frac{1}{2}(y')^2 + \frac{1}{2}y^2 \right] = -\varepsilon(y')^4 \leq 0. \quad (11.3.2)$$

Linear waves in higher dimensions

Fritz John, Chap 5, pg
simple model for
waves in acoustics
or optics

$$\square = \partial_t^2 - c^2 \Delta, \text{ D'Alembertian Operator}$$

\emptyset

$$\begin{aligned} \square u &= 0 \\ u(x_0) &= f(x) \\ u_t(x_0) &= g(x) \end{aligned} \quad \left. \right\} \quad t=0$$

(method due to Poisson)

PROPERTIES OF SPHERICAL MEANS

Let $h(x) = h(x_1, x_2, \dots, x_n) \in C(\mathbb{R}^n)$ with its spherical mean

defined by

\square_1

$$y = x + r\vec{z}$$

$$M_h(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} h(y) dS_y$$



$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

\square_2

or

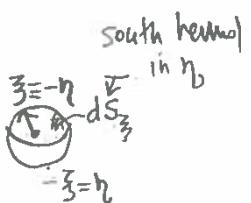
$$M_h(x, r) = \frac{1}{\omega_n} \int_{|\vec{z}|=1} h(x + r\vec{z}) dS_{\vec{z}}$$

\square_1 good def. for $r > 0$

\square_2 can extend to $r \in \mathbb{R}$. For ex. show M_h is even in r .

- With the later form we can easily see that M is an even function of r :

$$\begin{aligned} M_h(x, r) &= M_h(x_i - r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r(-\xi)) dS_\xi = \\ &= \frac{1}{\omega_n} \int_{|\eta|=1} h(x + r\eta) dS_\eta \end{aligned}$$



from
A11

let $h \in C^2(\mathbb{R}^n)$

$$\frac{\partial}{\partial r} M_h(x, r) = \frac{1}{\omega_n} \int_{|\xi|=1} \left(\sum_{i=1}^n h_{xi} \xi_i \right) dS_\xi =$$

normal vector to
the unit
sphere

$$\frac{\partial x_i}{\partial r} \xi_i$$

$$= \text{Div. Thm} = \frac{r}{\omega_n} \int_{|\xi| \leq 1} \Delta_x h(x + r\xi) d\xi =$$

$$= \frac{r}{\omega_n} \Delta x \int_{|\xi| \leq 1} h(x + r\xi) d\xi = \frac{r^n}{r^n \omega_n} \Delta x \int_{|y-x| \leq r} h(y) dy =$$

$$= \frac{r^{n-1} \Delta x}{\omega_n} \int_0^r dp \int_{|y-x|=p} h(y) dy = r^{n-1} \Delta x \int_0^r p^{n-1} \left[\frac{1}{\omega_n p^{n-1}} \int_{|y-x|=p} h(y) dy \right] dp.$$

Conclusion:

$$\frac{\partial}{\partial r} M_h(x, r) = \Delta x \int_0^r p^{n-1} M_h(x, p) dp$$

or even diff. to ge

$$\frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} M_h(x, r) = \Delta x \left(r^{n-1} M_h(x, r) \right)$$

① this leads to DARBOUX's equation

- no wave eqn. used.
- only properties of spher-mean

$$\square_2 \quad \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_h(x, r) = \Delta_x M_h(x, r)$$

$$\begin{cases} M_h(x, 0) = h(x) \\ \frac{\partial M_h}{\partial r}(x, 0) = 0 \end{cases}$$

(by continuity of h) \circledast
(since M is even in r)

$$\circledast \lim_{r \rightarrow 0} \frac{1}{\omega n r^{n-1}} \int_{|y-x|=r} h(y) dS_y = h(x)$$

— Now we finally go to the wave (D'Alembertian) equation, with $u(x, t)$ given by Φ . Namely take

$$M_u(x, r, t) = \frac{1}{\omega n} \int_{|\beta|=1} u(x+r\beta, t) dS_\beta$$

where $M_u(x, 0, t)$ gives back $u(x, t)$

Using \square_2 we have the Euler-Poisson-Darboux eq.:

$$\frac{\partial^2}{\partial t^2} M_u(x, r, t) = c^2 \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_u(x, r, t) \quad \begin{cases} M_u(x, 0, t) = M_f \\ \frac{\partial M_u}{\partial t}(x, 0, t) = M_g \end{cases}$$

We now have a scalar wave equation for the spherical mean,

the easiest case to solve is $n=3$, easier than $n=2$! Let $n=3$ and multiply

the Euler-Poisson-Darboux eq by r :

$$\frac{\partial^2}{\partial r^2} r M_u = c^2 \left(r \frac{\partial^2}{\partial r^2} + 2 \frac{\partial}{\partial r} \right) M_u$$

which yields

$$\Rightarrow \boxed{\frac{\partial^2}{\partial r^2} (r M_u) = c^2 \frac{\partial^2}{\partial r^2} (r M_u)}$$

a 1D wave equation for $(r M_u)$!

D'Alembert's solution:

$$\begin{aligned} r M_u(x, r, t) &= \frac{1}{2} \left[(r+ct) M_f(x, r+ct) + (r-ct) M_f(x, r-ct) \right] + \\ &+ \frac{1}{2c} \int_{r-ct}^{r+ct} \mathfrak{Z} M_g(x, \xi) d\xi, \end{aligned}$$

where the initial conditions are

$$\begin{cases} r M_u(x, r, 0) = r M_f(x, r) \\ \frac{\partial r M_u(x, r, 0)}{\partial t} = r M_g(x, r) \end{cases}$$

Using the fact that these are even functions, and preparing for \int_{-r}^r

$$\begin{aligned} M_u &= \frac{1}{2} \left[(ct+r) M_f(x, ct+r) - (ct-r) M_f(x, ct-r) \right] + \\ &+ \frac{1}{2c} \int_{ct-r}^{ct+r} \int_{r-ct}^{ct+r} \mathfrak{Z} M_g(x, \xi) d\xi dr \end{aligned}$$

$= 0$ (odd integrand)

see next page; why add this term in $\lim_{r \rightarrow 0}$.

$$\lim_{r \rightarrow 0} M_a(x_i, r) = \lim_{r \rightarrow 0} \left[\frac{(ct+r)M_f(x_i, ct+r) - (ct-r)M_f(x_i, ct-r)}{2r} \right] +$$

$$+ \lim_{r \rightarrow 0} \left[\frac{1}{2rc} \int_{ct-r}^{ct+r} \Im M_g(x_i, \tilde{s}) d\tilde{s} \right].$$

which equals

$$u(x_i, t) = \frac{\partial}{\partial r} (ct M_f(x_i, ct)) + \frac{1}{c} (ct M_g(x_i, ct))$$

$$u(x_i, t) = \frac{\partial}{\partial t} (t M_f(x_i, ct)) + t M_g(x_i, ct)$$

Kirchoff formula.

And going back to the original functions:

$$u(x_i, t) = \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} g(y) dS_y + \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_{|y-x|=ct} f(y) dS_y \right]$$

3D
Soln.

Remark: both terms in $\textcircled{1}$, satisfy the linear wave eqn.

the solution can also be written in the form

call h

$$u(x_i, t) = \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \left[g(y) + f(y) + \sum_i f_{yj}(y) (y_j - x_i) \right] dS_y = M_h(x_i, ct)$$

2

179)

$$h(y; x_i, t) = \text{tg}(y) + f(y) + \sum_i f_{y_i}(y_i - x_i)$$

Even though caustics can form (depending on the initial data) ...

... the energy is conserved:

$$E(t) = \frac{1}{2} \int_{\Omega} \left(u_t^2(x, t) + c^2 \sum u_{x_i}^2(x, t) \right) dx$$

Check:

$$\begin{aligned} \frac{dE}{dt} &= \int_{\Omega} \left(u_t u_{tt} + c^2 \sum u_{x_i} u_{x_i tt} \right) dx = \\ &= \int_{\Omega} \left(u_t u_{tt} + c^2 \sum (u_t u_{x_i})_{x_i} \right) dx \quad \leftarrow \text{a term was borrowed} \end{aligned}$$

When $u(x, t) = 0$ for $|x|$ large then
(u_{x_i} also = 0)

$$\boxed{\frac{dE}{dt} = 0}$$

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{|\mathbf{y}-\mathbf{x}|=ct} h(\mathbf{y}; \mathbf{x}, t) dS_y$$

Fig.1

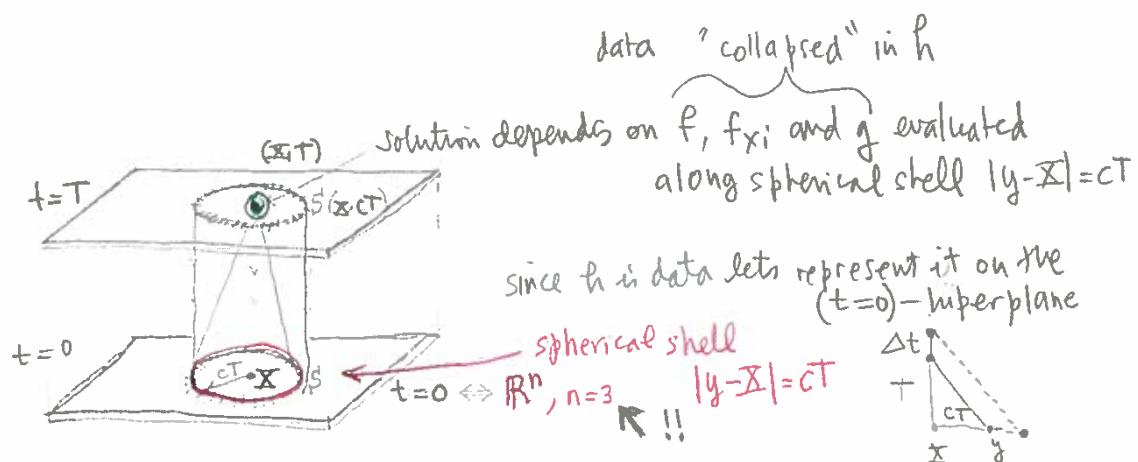


Fig.2

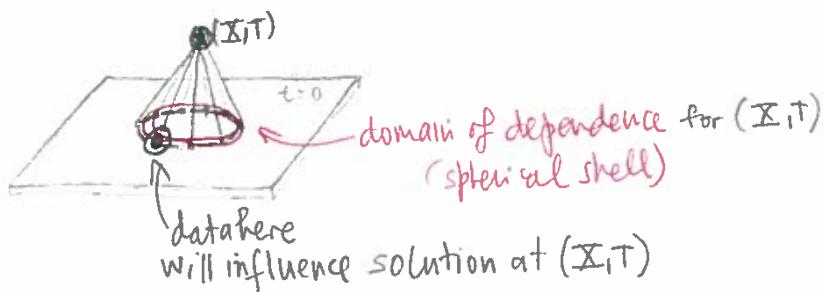
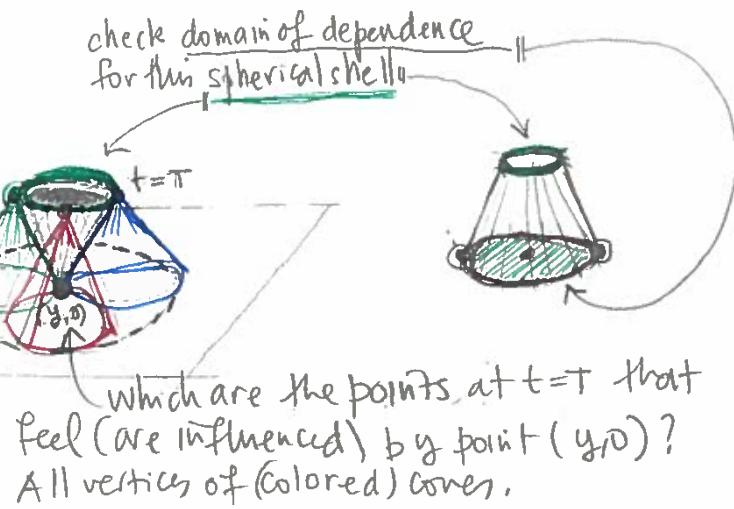
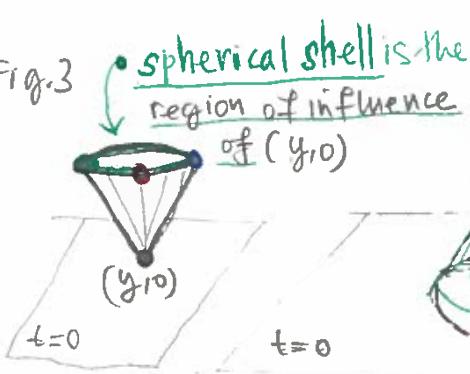
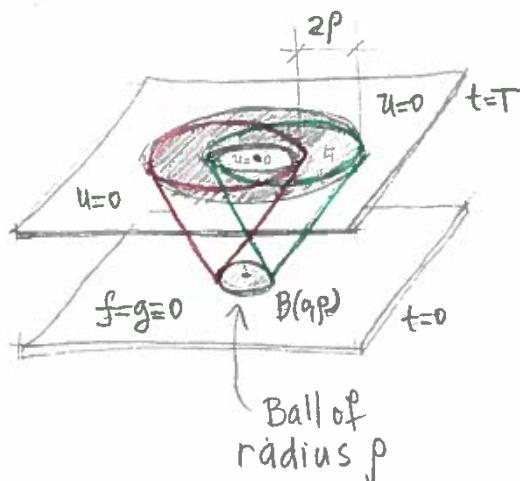
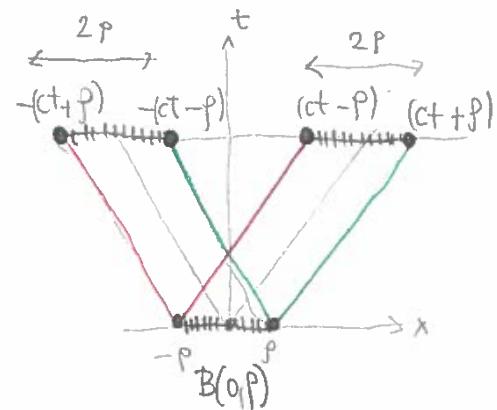


Fig.3



Since domain of dependence at (x, t) is a surface (rather than a solid region) ...



$$\begin{aligned} S_1 &= S_1'(0, ct-p) \\ S_2 &= S_2'(0, ct+p) \end{aligned}$$

$u=0, t > |x| + p$
 When $x \rightarrow \frac{|x|+p}{c}$

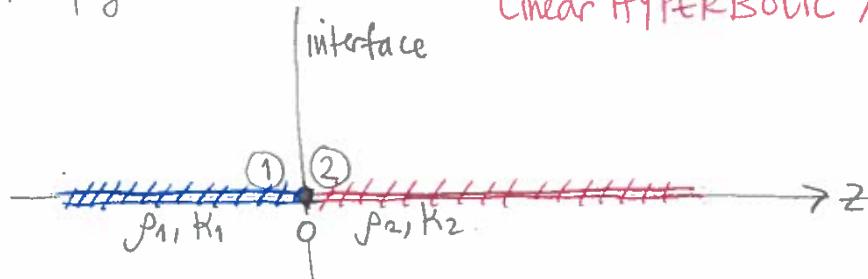
$$u=0 \text{ in } |x| < ct-p$$

... we get Huygens principle. "This accounts for the possibility of sharp signals being transmitted in 3D". After a while the disturbance is NOT felt any more at (x, t) . Since area increases, due to spreading u decreases as $t \uparrow$.

$n=2$

get solution from Hadamard's method of descent. Namely look a $n=3$ solutions that do not depend on x_3 , $x = (x_1, x_2, x_3)$.

REFLECTION-TRANSMISSION @ INTERFACE
Linear HYPERBOLIC ACOUSTIC WAVE



$$\left\{ \begin{array}{l} \rho(z) u_t + p_z = 0 \\ \frac{1}{K(z)} p_t + u_z = 0 \end{array} \right. \quad c(z) = \sqrt{K/\rho}, \quad S(z) = \sqrt{K\rho}$$

$$\begin{aligned} A(z_1 t) &= S^{-1/2} p(z_1 t) + S^{1/2} u(z_1 t) & A_1(0, t) &= f(t) \text{ incoming } \\ B(z_1 t) &= -S^{-1/2} p(z_1 t) + S^{1/2} u(z_1 t) & B_2(0, t) &= 0 \text{ no incoming } \end{aligned}$$

$$\left\{ \begin{array}{l} A_t + c A_z = -r B \\ B_t - c B_z = r A \end{array} \right. \quad r(z) = \frac{S'}{2S} = \frac{d}{dz} \log S(z)$$

reflectivity

Recall issues of PSet 2
and equations with
discontinuous coefficients

Continuity of p and u at interface

$$u(0, t) = S_1^{-1/2} \left(\frac{A_1(0, t) + B_1(0, t)}{2} \right) = S_2^{-1/2} \left(\frac{A_2 + B_2}{2} \right)$$

$$p(0, t) = S_1^{-1/2} \left(\frac{A_1 - B_1}{2} \right) = S_2^{-1/2} \left(\frac{A_2 - B_2}{2} \right)$$

$$S_1^{-1/2} A_1 + S_1^{-1/2} B_1 = S_2^{-1/2} A_2 + S_2^{-1/2} B_2$$

$$S_1^{1/2} A_1 - S_1^{1/2} B_1 = S_2^{1/2} A_2 - S_2^{1/2} B_2$$

$$\begin{bmatrix} S_1^{-1/2} & S_1^{-1/2} \\ S_1^{1/2} & -S_1^{1/2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} S_2^{-1/2} & S_2^{-1/2} \\ S_2^{1/2} & -S_2^{1/2} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

$$\begin{bmatrix} 1/a & 1/a \\ a & -a \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} a & 1/a \\ a & -1/a \end{bmatrix}$$

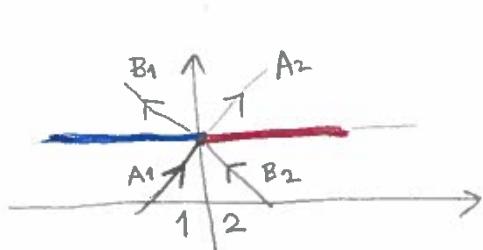
$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} S_2^{1/2} & -S_2^{-1/2} \\ S_2^{-1/2} & -S_2^{1/2} \end{bmatrix} \begin{bmatrix} S_1^{-1/2} & S_1^{-1/2} \\ S_1^{1/2} & -S_1^{1/2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

⊕ $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (S_2/S_1)^{1/2} + (S_1/S_2)^{1/2} & (S_2/S_1)^{1/2} - (S_1/S_2)^{1/2} \\ (S_2/S_1)^{1/2} - (S_1/S_2)^{1/2} & (S_2/S_1)^{1/2} + (S_1/S_2)^{1/2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$

propagator matrix $P = \begin{bmatrix} p^+ & p^- \\ p^- & p^+ \end{bmatrix}$, $p^\pm = \frac{(S_2/S_1)^{1/2} \pm (S_1/S_2)^{1/2}}{2}$

Converts modes from one side of the interface to modes on the other side.

$$\det P = p^{+2} - p^{-2} = \\ = \frac{[(S_2/S_1) + 2 + (S_1/S_2)] - [(S_2/S_1) - 2 + (S_1/S_2)]}{4} =$$



are in \oplus $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = P \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$

$$\begin{bmatrix} A_2 \\ 0 \end{bmatrix} = P \begin{bmatrix} f(t) \\ B_1 \end{bmatrix}$$

Conservation: $R^2 + T^2 = 1$

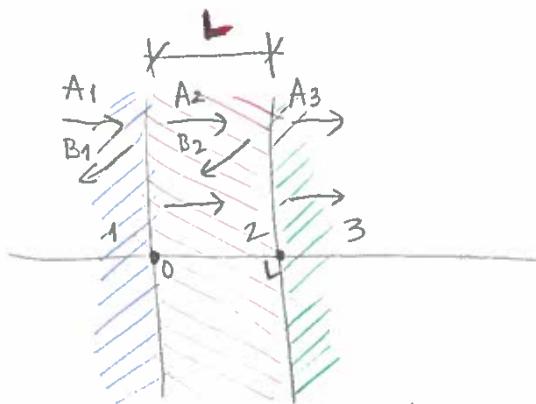
Data

$$\left\{ \begin{array}{l} A_1(0,t) = f_1(t) \text{ incoming wave} \\ B_1(0,t) = ? \\ A_2(0,t) = ? \\ B_2(0,t) = 0 \end{array} \right.$$

reflection co. $B_1 = R = -\frac{p^-}{p^+} = \frac{S_1}{S_2}$

$$\left\{ \begin{array}{l} A_2 = p^+ f + p^- B_1 \\ 0 = p^- f + p^+ B_1 \end{array} \right.$$

transmission Coeff $A_2 = T = \frac{1}{2} \frac{1}{p^+} = \frac{2\sqrt{S_1 S_2}}{S_1 + S_2}$



For multi-layers use: FGP5 page 50

$$A_j(t, z) = a_j(t - (z - L_{j-1}) / c_j)$$

$$B_j(t, z) = b_j(t + (z - L_{j-1}) / c_j)$$

travel time
in the j-layer

"ringing" will generate delays.

In Fouque, Garnier, Papamichael & Sola (page 42)

we have that this case is best to be studied in Fourier space

$$\begin{bmatrix} \hat{A}_3 \\ 0 \end{bmatrix} = \hat{P}(w) \begin{bmatrix} \hat{f}(w) \\ \hat{B}_1(w) \end{bmatrix}$$

$$\begin{bmatrix} A_2(0, t) \\ B_2(0, t) \end{bmatrix} = P_1 \begin{bmatrix} f(t) \\ B_1 \end{bmatrix}$$

$$\begin{bmatrix} A_3(L, t) \\ B_3(L, t) \end{bmatrix} = P_2 \begin{bmatrix} A_2(0, t-L/c_2) \\ 0 \end{bmatrix}$$

propagator in freq. domain.

$$\hat{P} = \hat{P}_2 \hat{P}_1$$

$$\hat{P}_1 = \begin{bmatrix} p^+ & p^- \\ p^- & p^+ \end{bmatrix}$$

$$\hat{P}_2 = \begin{bmatrix} p e^{i w L / c_2} & p e^{-i w L / c_2} \\ p e^{i w L / c_2} & p e^{-i w L / c_2} \end{bmatrix}$$

In the frequency domain one can deal with delays as phase factors.

Remark:

For some results the Gouffiland medium is used \rightarrow

\rightarrow uniform travel time in random layers.

At the end	$\hat{B}_1(w) = \hat{P}(w) \hat{f}(w)$	$\hat{A}_3(w) = \hat{T}(w) \hat{f}(w)$
------------	--	--

$\hat{R}(w) = \frac{p_2 e^{2 i w L / c_2} + R_1}{1 + R_1 R_2 e^{2 i w L / c_2}}$	$\hat{T}(w) = \frac{T_1 T_2 e^{i w L / c_2}}{1 + R_1 R_2 e^{2 i w L / c_1}}$
--	--

$$R_1 = \frac{S_1 - S_2}{S_1 + S_2}, R_2 = \frac{S_2 - S_3}{S_2 + S_3}, T_1 = \frac{2 \sqrt{S_1 S_2}}{S_1 + S_2}, T_2 = \frac{2 \sqrt{S_2 S_3}}{S_2 + S_3}$$